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Construction of New Delay-Tolerant Space-Time Codes

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***Abstract*—Perfect Space-Time Codes (STC) are full-rate, full-diversity codes originally proposed for Multiple Input Multiple Output (MIMO) systems. Based on Cyclic Division Algebras (CDA), they have non-vanishing determinants and hence achieve the Diversity-Multiplexing Tradeoff (DMT). In addition, these codes have led to optimal distributed Space-Time Codes when applied in cooperative networks under the assumption of perfect synchronization between relays. However, they lose their diversity when delays are introduced and thus are not delay-tolerant. In this paper, using the cyclic division algebras of perfect codes, we construct new codes that maintain the same properties as perfect codes in the synchronous case. Moreover, these codes preserve their full-diversity in asynchronous transmission.**

I. Problem Statement

During the past decade, cooperative diversity has emerged as a new form of spatial diversity in wireless communication systems. It counteracts the need of incorporating many antennas into a single terminal, especially in cellular systems and ad-hoc sensor networks, where it can be impractical for a mobile unit to carry multiple antennas due to its size, power and cost limitations. In cooperative networks, users communicate cooperatively to transmit their information by using distributed antennas belonging to other independent terminals. A virtual MIMO scheme is then created, where a transmitter is also acting as a relay terminal, with or without some processing, assisting another transmitter to convey its messages to a destination. In order to achieve the cooperative diversity and the full data rate, space-time coding techniques of MIMO systems have also been applied yielding many designs of distributed STC under the assumption of synchronized relay terminals. However, this *a priori* condition on synchronization can be quite costly in terms of signaling and even hard to handle in relay networks [1]. Unlike conventional MIMO transmitter, equipped with one antenna array using one local oscillator, distributed antennas are dispersed on different terminals, each one with its local oscillator. Thus, they are not sharing the same timing reference, resulting in an asynchronous cooperative transmission.

On the other hand, in a synchronous transmission, the distributed space-time codes are constructed basically according to the rank and determinant criteria [2] and hence aim at achieving full diversity. Note that the rows of the codeword matrix represent the different relay terminals (antennas). So, when asynchronicity is evoked, delays are introduced between transmitted symbols from different distributed antennas shifting the matrix rows. This matrix misalignment can cause rank deficiency of the space-time code, and thus performance degradation. Therefore, the codes previously designed are no more effective unless they tolerate asynchronicity. Furthermore, an efficient code design should satisfy the full-diversity order for any delay profile. This intends to guarantee full-rank codewords distance matrix *i.e.,* its rank equal to the number of involved relays, leading to the so-called delay-tolerant distributed space-time codes.

Such codes were firstly designed in [1] as full-diversity binary Space-Time Trellis codes (STTC) based on the stacking construction by Hammons and El Gamal, its generalization to Lu-Kumar multilevel space-time codes, and the extension of the latter codes for more diverse AM-PSK constellations [3], [4]. Systematic construction including the shortest STTC with minimum constraint length was also proposed in [5], as well as some delay-tolerant short binary Space-Time Block Codes (STBC) [6]. Recently, Damen and Hammons extended the Threaded Algebraic Space-Time (TAST) codes to asynchronous transmission [7]. The delay-tolerant TAST codes are based on three different thread structures where the threads are separated by using different algebraic or transcendental numbers that guarantee a non-zero determinant of the codewords distance matrix. An extension of this TAST framework to minimum delay length codes was considered in [8].

Meanwhile, perfect space-time block codes originally constructed for MIMO systems [9]–[11], were also investigated for wireless relay networks. Optimal coding schemes in the sense of DMT tradeoff [12], based code division algebras, were provided for any number of users and for different cooperative strategies. Nevertheless, all these schemes assumed perfect synchronization between users. Then, it was in [13] that Petros and Kumar discussed the delay-tolerant version of the optimal perfect code variants for asynchronous transmission. They stated that delay-tolerant diagonally-restricted CDA codes and delay-tolerant full-rate CDA codes can be obtained from previous designs by multiplying the codeword matrix by a random unitary matrix. This matrix can be taken specifically from an infinite set of unitary matrices that do not have elements in the code field.

In this paper, we construct delay-tolerant distributed codes based on the perfect codes algebras from a different point of view. The new construction is obtained from the tensor product of two cyclic division algebras, one of them being the perfect code CDA. The codes are designed in such a way to maintain the same properties of their corresponding perfect codes in the synchronous transmission, namely full-rate, full-diversity and non-vanishing minimum determinant (NVD). In addition, unlike the perfect codes, the new codes preserve the full-diversity in the asynchronous transmission. In [14], we have used this approach to design a delay-tolerant 2×2 code based on the Golden Code CDA. Here, we detail the general construction and we propose new delay-tolerant 3×3 and 4×4 codes.

In the sequel, the paper is organized as follows. The cooperative system model is introduced in section II. In section III, the general construction is described and the new delay-tolerant codes are designed. Some simulation results are shown in section IV and finally, the conclusion is given in section V.

II. Cooperative System Model

A. System Description

We consider a cooperative system with a source S communicating to a destination D via M relays R_i in two phases, and without direct links between the source and the destination. In the first phase, the source broadcasts its message to the potential relays. In the second phase, the relays use the Decode-and-Forward protocol to detect the source message then if successfully detected transmit it to the destination. We assume that all the M relays are able to achieve error free decoding. Since the relays transmission overlap in time and frequency, they can cooperatively implement a distributed space-time code. Considering only the second phase of transmission, the system is equivalent to a MIMO scheme where the distributed $M \times T$ perfect space-time code $\mathcal{P}$ is used by the relays, with M transmit antennas one by relay, and N_r receive antennas at the destination. Every time slot $t, t = 1 \dots T$, the relays send the $M \times 1$ t^{th} column vector $\mathbf{X}_t$ of the codeword $\mathbf{X}$ and the destination receives

$$\mathbf{Y}_t = \mathbf{H}_t\mathbf{X}_t + \mathbf{W}_t, \quad \mathbf{Y}_t, \mathbf{W}_t \in \mathbb{C}^{N_r\times 1} \tag{1}$$

$\mathbf{H}_t$ is the $N_r \times M$ complex channel matrix and $\mathbf{W}_t$ is the complex white Gaussian noise. The channel is assumed quasi-static with constant fadings during a transmitted codeword and independent fadings between subsequent codewords. Dealing with square space-time codes ($M = T$), the codeword matrix contains M^2 information symbols $s_1, \dots, s_{M^2}$ carved from two-dimensional QAM or HEX finite constellations $\mathcal{S}$. These constellations are chosen from the ring of Gaussian integers $\mathbb{Z}[i]$ or Eisenstein integers $\mathbb{Z}[j]$ $(i = \sqrt{-1}, j = e^{2\pi i/3})$.

B. Asynchronous Cooperative Diversity

The above expression (1) is valid only when relays are synchronized. In the presence of asynchronicity, the codeword transmission is spanned on more than T symbol intervals due to delays. We assume that the relays are synchronized at the frame-codeword level and hence there is no interference between codewords transmission. We further assume that the timing errors between different relays are integer multiples of the symbol duration and the fractional timing errors are absorbed in the channel dispersion. In the codeword matrix, the delays are filled with zeros; they are known at the receiver but not at the transmitting relays [1].

Denoting a delay profile by $\mathfrak{d} = (\mathfrak{d}_1, \mathfrak{d}_2, \dots, \mathfrak{d}_M)$, a delay $\mathfrak{d}_i$ corresponds to the relative delay of the received signal from the i^{th} relay as referenced to the earliest received relay signal. Let $\mathfrak{d}_{\max}$ denotes the maximum of the relative delays, then from the receiver perspective, the $M \times (T + \mathfrak{d}_{\max})$ codeword matrix was sent instead of the $M \times T$ space-time code.

The diversity order of any space-time code is defined by the minimum rank of the distance codeword matrix over all pairs of distinct codewords [2]. The distributed $M\times T$ perfect codes are full-rate full-diversity. However, when asynchronicity is introduced, certain delay profiles $\mathfrak{d}$ can result in linearly-dependent rows, thus the code will lose its full-diversity. Therefore, we aim to construct new codes that are based on the perfect codes CDAs such that they maintain the same properties as perfect codes in the synchronous case. But also, these codes preserve their full-diversity in asynchronous transmission, hence are delay-tolerant for arbitrary delay profile.

III. Construction of Delay-Tolerant Distributed Codes based Perfect Codes algebras

A. General Construction

The approach consists in constructing a division algebra isomorphic to the tensor product (also called Kronecker product or cross-product) of two cyclic division algebras of lower degrees. Previous constructions based on the cross-product algebras have been investigated, for instance in [11], [15], [16] either for prime or coprime degrees of the composite algebras. In these constructions, the space-time code was build on the cyclic product algebra. However, in the present construction, the higher degree algebra is only used to derive appropriately the space-time code.

Since we intend to construct a full-rate $M \times M$ STC that is based on the CDA of the full-rate $M \times M$ perfect code, then the first algebra to be considered is the cyclic division algebra of the perfect code $\mathcal{A}_1(\mathbb{K}_1/\mathbb{F}, \sigma_1, \gamma_1)$ of degree $n_1 = M$ over the base field $\mathbb{F} = \mathbb{Q}(i)$ or $\mathbb{F} = \mathbb{Q}(j)$. Recalling the perfect code construction [9], [10], $\mathbb{K}_1 = \mathbb{Q}(i, \theta_1)$ or $\mathbb{K}_1 = \mathbb{Q}(j, \theta_1)$ is an extension of degree M over $\mathbb{F} = \mathbb{Q}(i)$ or $\mathbb{F} = \mathbb{Q}(j)$, respectively. θ_1 is an algebraic number, σ_1 is the generator of the cyclic Galois group of $\mathbb{K}_1$ $\mathrm{Gal}(\mathbb{K}_1/\mathbb{F}) = \langle\sigma_1\rangle$ and γ_1 a non-norm element suitable for the cyclic extension $\mathbb{K}_1/\mathbb{F}$.

For sake of simplicity, we analyze in the sequel the case of Gaussian Field $\mathbb{Q}(i)$ to explain the construction. The principal ideal $\mathcal{I}_{\mathbb{K}_1} \subseteq \mathcal{O}_{\mathbb{K}_1}$ is generated by an element α and its integral basis is $\mathrm{B}_1 = (v_1, v_2, \dots, v_M)$ (or if unitary $\mathrm{B}_1 = (\alpha, \alpha\theta_1, \dots, \alpha\theta_1^{M-1})$). The basis of the complex algebraic lattice $\Lambda(\mathcal{I}_{\mathbb{K}_1})$ is obtained by applying the canonical embedding to B_1. Consequently, the generator matrix corresponds to the unitary rotation matrix in $\mathbb{Z}[i]^M$

$$\mathbf{M}_1 = \frac{1}{\sqrt{p_1}}\begin{bmatrix} v_1 & v_2 & \dots & v_M \\ \sigma_1(v_1) & \sigma_1(v_2) & \dots & \sigma_1(v_M) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_1^{M-1}(v_1) & \sigma_1^{M-1}(v_2) & \dots & \sigma_1^{M-1}(v_M) \end{bmatrix}, \tag{2}$$

with $\sqrt{p_1}$ the normalization factor.

Now, we consider another Galois extension $\mathbb{K}_2$ over $\mathbb{F}$ of the same degree $n_2 = M$ such that its discriminant is coprime to the one of $\mathbb{K}_1$ i.e., $(d_{\mathbb{K}_1}, d_{\mathbb{K}_2}) = 1$. Let $\mathbb{K}_2 = \mathbb{Q}(\theta_2)$ with θ_2 an algebraic number. The Galois group is generated by σ_2 as $\mathrm{Gal}(\mathbb{K}_2/\mathbb{F}) = \langle\sigma_2\rangle$. The principal ideal of the algebra is such that $\mathcal{I}_{\mathbb{K}_2} = \mathcal{O}_{\mathbb{K}_2}$ and thus its integral basis is given by $\mathrm{B}_2 = (1, \theta_2, \ldots, \theta_2^{M-1})$. The canonical embedding of B_2 gives another complex rotated lattice of $\mathbb{Z}[i]^M$ that is generated by the unitary matrix $\mathbf{M}_2$ with $\sqrt{p_2}$ the normalization factor,

$$\mathbf{M}_2 = \frac{1}{\sqrt{p_2}} \begin{bmatrix} 1 & \theta_2 & \ldots & \theta_2^{M-1} \\ 1 & \sigma_2(\theta_2) & \ldots & \sigma_2(\theta_2^{M-1}) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \sigma_2^{M-1}(\theta_2) & \ldots & \sigma_2^{M-1}(\theta_2^{M-1}) \end{bmatrix}. \quad (3)$$

The tensor product of both algebras allows to build a rotated lattice in higher dimension corresponding to the complex $M^2 \times M^2$ unitary matrix $\mathbf{M}$ based on the previous $M \times M$ constructions. According to [15],

Proposition 1: : Let $\mathbb{K}$ be the compositum of the above Galois extensions, $\mathbb{K} = \mathbb{K}_1\mathbb{K}_2 = \mathbb{Q}(i, \theta_1, \theta_2)$ of order $n = n_1 n_2 = M^2$ over $\mathbb{F}$. Since $\mathbb{K}_1$ and $\mathbb{K}_2$ have coprime discriminants, the corresponding lattice generator matrix can be obtained as the tensor product of the previous unitary generator matrices $\mathbf{M} = \mathbf{M}_2 \otimes \mathbf{M}_1$.

Consequently,

Proposition 2: : Let $m_j = [\mathbb{K} : \mathbb{K}_j] = n/n_j,\ j = 1, 2$ the order of the extensions, then the discriminant of $\mathbb{K}$ is $d_{\mathbb{K}} = d_{\mathbb{K}_1}^{m_1} d_{\mathbb{K}_2}^{m_2}$. The minimum product distance of the lattice is derived from the discriminant of $\mathbb{K}$ as

$$d_{\mathrm{p,min}} = \frac{1}{\sqrt{d_{\mathbb{K}}}} = \frac{1}{\sqrt{d_{\mathbb{K}_1}^{m_1} d_{\mathbb{K}_2}^{m_2}}}. \quad (4)$$

Using the matrix $\mathbf{M}$, the space-time coded components are given by the linear combination $\mathbf{x} = \mathbf{M}\mathbf{s}$ where $\mathbf{s} = [s_1, s_2, \ldots, s_{M^2}]^T$ is the information symbol vector carved from a q-QAM constellation ($\in \mathbb{Z}[i]$). Then, the space-time codeword matrix is defined by distributing the components with appropriate constant factors $\phi_l, l = 1, \ldots, M^2$. It can be represented as a Hadamard product

$$\mathbf{X} = [\Phi] \bullet [\mathbf{x}] = \begin{bmatrix} \phi_1 x_1 & \cdots & \phi_{M(M-1)+1} x_{M(M-1)+1} \\ \phi_2 x_2 & \cdots & \phi_{M(M-1)+2} x_{M(M-1)+2} \\ \vdots & \ddots & \vdots \\ \phi_M x_M & \cdots & \phi_{M^2} x_{M^2} \end{bmatrix} \quad (5)$$

The key idea in the code construction is to determine the coefficients ϕ_l that allow to preserve the same properties of the corresponding perfect codes $\mathcal{P}$ in synchronous transmission.

- On one side, it can be seen that the new code transmits M^2 information symbols and has full-rate $R = M$ spcu.
- On the other side, the rank criterion tells that for perfect codes the full-diversity is obtained if the determinant of the difference of two distinct codewords of $\mathcal{P}$ is non-zero. For linear codes, this condition is reduced to

$$\det(\mathbf{X}\mathbf{X}^\dagger) = |\det(\mathbf{X})|^2 \neq 0, \quad \mathbf{X} \neq \mathbf{0}, \mathbf{X} \in \mathcal{P} \quad (6)$$

  So we need to find the ϕ_l factors that satisfy this condition in order to have full-diversity codes.
- Moreover, the perfect codes have non-vanishing minimum determinants. Then, we are interested in deriving $M \times M$ STCs that have not only non-zero determinants, but also these determinants do not vanish when constellation size increases.
- In order to guarantee uniform energy distribution in the codeword, we ask that $|\phi_l| = 1$. Choosing further the coefficients $\phi_l \in \mathcal{O}_{\mathbb{F}} = \mathbb{Z}[i]$ yields better determinants as obtained for the non-norm elements γ of the perfect codes [10]. This restricts the values of ϕ_l to $\phi_l = \pm 1, \pm i$.
- It can also be noticed that the new code satisfies the cubic shaping property since the generator matrix $\mathbf{M}$ of the M^2-dimensional lattice is unitary, and hence the code is information lossless.

In addition, when asynchronicity between relays is involved, the rank criterion should be verified for the shifted matrix. Another criterion will be analyzed that is the non-zero product distance of the codeword matrix in order to prove that the new codes are delay-tolerant.

B. 2×2 Code based on Golden Code

In [14], we proposed a new 2×2 delay-tolerant code based on the Golden Code. This code follows the previous approach for dimension $M = 2$. The 4×4 unitary matrix is obtained by the tensor product algebras of degree 4

$$\mathbf{M}_4 = \frac{1}{\sqrt{10}} \begin{bmatrix} \alpha & \alpha\theta_1 & \alpha\zeta_8 & \alpha\theta_1\zeta_8 \\ \bar{\alpha} & \bar{\alpha}\bar{\theta_1} & \bar{\alpha}\zeta_8 & \bar{\alpha}\bar{\theta_1}\zeta_8 \\ \alpha & \alpha\theta_1 & -\alpha\zeta_8 & -\alpha\theta_1\zeta_8 \\ \bar{\alpha} & \bar{\alpha}\bar{\theta_1} & -\bar{\alpha}\zeta_8 & -\bar{\alpha}\bar{\theta_1}\zeta_8 \end{bmatrix} \quad (7)$$

and the codeword matrix is defined by

$$\mathbf{\Gamma}(\mathbf{s}) = \begin{bmatrix} \phi_1 x_1 & \phi_3 x_3 \\ \phi_2 x_2 & \phi_4 x_4 \end{bmatrix}, \quad (8)$$

where x_i are the components of the vector $\mathbf{M}_4\mathbf{s}$ with s_1, s_2, s_3, s_4 are q-QAM symbols. By choosing $\phi_1 = \phi_3 = 1$ and $\phi_2 = \phi_3 = i$, we showed in [14] that the determinant of the new code is equal to the Golden code determinant $\delta_{\min}(\mathcal{C}) = 1/5$, and does not vanish when increasing the size of the QAM constellation. Here, we can also notice that the coefficients ϕ_l can be changed equivalently to the coefficients of the Fourier matrix $F_M = (w^{jk})$ where $w = e^{2i\pi/M}$ is the primitive $M^{th} = 2^{nd}$ root of unity with $\phi_1 = \phi_3 = \phi_4 = 1$ and $\phi_2 = -1$. As a direct consequence from the tensor product construction, equation (4) gives $d_{\mathrm{p,min}} = 1/\sqrt{d_{\mathbb{K}}} = 1/\sqrt{5^2 4^2} = 1/20$. Thus, the minimum product distance $d_p = \prod_{j=0}^{4} |x_j|$ is non-zero and the shifted matrix is full rank unless $\mathbf{s} = \mathbf{0}$. Therefore, the new code unlike the Golden code keeps its full-diversity in the case of asynchronous relays.

C. 3×3 Code based on 3×3 Perfect Code

In order to construct now the delay-tolerant 3×3 code, we consider the base field $\mathbb{F} = \mathbb{Q}(j)$ and we use q-HEX symbols. Let $\theta_1 = \zeta_7 + \zeta_7^{-1} = 2\cos(\frac{2\pi}{7})$, with ζ_7 the 7^{th} root of unity. The 3×3 perfect code was constructed using the cyclic division algebra $\mathcal{A}_1(\mathbb{K}_1/\mathbb{F},\sigma_1,j)$ of order 3 [10], with $\mathbb{K}_1 = \mathbb{Q}(j,\theta_1)$ the relative extension and $\mathrm{Gal}(\mathbb{K}_1/\mathbb{F}) = \langle\sigma_1\rangle$, $\sigma_1 : \zeta_7+\zeta_7^{-1} \mapsto \zeta_7^2+\zeta_7^{-2}$. The corresponding complex lattice $\Lambda(\mathcal{I}_{\mathbb{K}_1})$ is a rotated version of $\mathbb{Z}[j]^3$ and it is generated by

$$\mathbf{M}_1 = \frac{1}{\sqrt{7}}\begin{bmatrix} v_1 & v_2 & v_3 \\ \sigma_1(v_1) & \sigma_1(v_2) & \sigma_1(v_3) \\ \sigma_1^2(v_1) & \sigma_1^2(v_2) & \sigma_1^2(v_3) \end{bmatrix}. \quad (9)$$

with $\{v_k\}_{k=1}^3 = \{(1+j)+\theta_1, (-1-2j)+j\theta_1^2, (-1-2j)+(1+j)\theta_1+(1+j)\theta_1^2\}$. The relative discriminant of $\mathbb{K}_1$ is $d_{\mathbb{K}_1}=49$. Another extension of $\mathbb{F}$ of degree 3 that has coprime discriminant with $\mathbb{K}_1$ is the cyclotomic extension $\mathbb{K}_2=\mathbb{Q}(\zeta_9)$ with $\zeta_9=e^{2i\pi/9}$ the primitive 9^{th} root of unity and $d_{\mathbb{K}_2}=27$. Its Galois group $\mathrm{Gal}(\mathbb{K}_2/\mathbb{F})$ is generated by $\sigma_2 : \zeta_9 \mapsto j\zeta_9$. The integral basis of $\mathbb{K}_2$ is $\mathrm{B}_2=(1,\zeta_9,\zeta_9^2)$ and the lattice generator matrix is

$$\mathbf{M}_2 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & \zeta_9 & \zeta_9^2 \\ 1 & j\zeta_9 & j^2\zeta_9^2 \\ 1 & j^2\zeta_9 & j\zeta_9^2 \end{bmatrix}. \quad (10)$$

The compositum of both extensions $\mathbb{K}=\mathbb{K}_1\mathbb{K}_2=\mathbb{Q}(j,2\cos(\frac{2\pi}{7}),\zeta_9)$ is of order 9 over $\mathbb{Q}(j)$. Then, the corresponding 9-dimensional complex lattice is generated by the 9×9 unitary matrix $\mathbf{M}_9=\mathbf{M}_2\otimes\mathbf{M}_1$ and the 3×3 space-time code is defined by the matrix

$$\mathbf{\Gamma}(\mathbf{s}) = \begin{bmatrix} \phi_1x_1 & \phi_4x_4 & \phi_7x_7 \\ \phi_2x_2 & \phi_5x_5 & \phi_8x_8 \\ \phi_3x_3 & \phi_6x_6 & \phi_9x_9 \end{bmatrix}, \quad (11)$$

where x_i are the components of vector $\mathbf{M}_9\mathbf{s}$ and $\mathbf{s}$ the information symbol vector carved from q-HEX constellation.

1) Non-vanishing minimum determinant: By proceeding as previously, we need to determine the coefficients $\phi_l, l=1,\dots,9$ that guarantee the NVD property. In order to get $|\phi_l|=1$ so that a uniform average energy is transmitted per antenna, and to obtain better values of the determinant, we limit the choice of ϕ_l to $\phi_l=\pm1,\pm j$. By developing the code determinant with symbolic computation under Mathematica, we find that it has the same expression as the 3×3 perfect code determinant by choosing ϕ_l as the Fourier matrix coefficients in $\mathbb{Q}(j)$

$$\Phi = \begin{bmatrix} 1 & 1 & 1 \\ 1 & j & j^2 \\ 1 & j^2 & j \end{bmatrix}. \quad (12)$$

Therefore, the 3×3 infinite code $\mathbf{\Gamma}(\mathbf{s})$ has non-vanishing minimum determinant equal to $\delta_{\min}(\mathcal{C}) = 1/d_{\mathbb{K}_1} = 1/49$.

2) Delay-tolerance: On the other hand, to prove the delay-tolerance of this code, we should guarantee that the corresponding shifted codeword matrices are full rank. Therefore, it suffices to verify that for each asynchronous matrix there exists a square 3×3 matrix that is full rank *i.e.,* its determinant is non-zero. In fact, if we enumerate all delay profiles, it can be noticed that the problem of guaranteeing full-rank shifted matrices turns to guarantee that

- All component products $\subseteq d_{\mathrm{p}}=\prod_{i=1}^9|x_i|$ are non-zero that is always verified since the product distance $d_{\mathrm{p}}\neq0$ over $\mathbb{Z}[j]$ as $d_{\mathrm{p,min}}=1/\sqrt{49^3 27^3}$.
- All 2×2 minors of $\mathbf{\Gamma}(\mathbf{s})$ are non-zero that is equivalent to verify that the 9 entries of the cofactor matrix of $\mathbf{\Gamma}$ are non-zero.

Sketch of the proof:
In order to prove the second condition, we find two unitary matrices $\mathbf{U}$ and $\mathbf{V}$ such that the codeword matrix $\mathbf{\Gamma}$ can be written as $\mathbf{\Gamma}=\mathbf{UZV}$ for all $\mathbf{s}$, with $\mathbf{Z}$ the perfect code matrix and $\mathbf{U}$ and $\mathbf{V}$ defined by

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & j^2\zeta_9^2 & 0 \\ 0 & 0 & j^2\zeta_9 \end{bmatrix}, \ \mathbf{V} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ \zeta_9 & j\zeta_9 & j^2\zeta_9 \\ \zeta_9^2 & j^2\zeta_9^2 & j\zeta_9^2 \end{bmatrix}. \quad (13)$$

Let define the cofactor matrix of the perfect code by $\widetilde{\mathbf{Z}}$. Since $\mathbf{Z}$ is a finite subset of the cyclic division algebra $\mathcal{A}_1$, $\widetilde{\mathbf{Z}}$ is also a subset of $\mathcal{A}_1$ taken from the lattice $\Lambda=\mathcal{O}_{\mathbb{K}_1}\oplus e\mathcal{O}_{\mathbb{K}_1}\oplus e^2\mathcal{O}_{\mathbb{K}_1}$ with $e^3=j$ and $\mathcal{O}_{\mathbb{K}_1}$ the ring of integers of $\mathbb{K}_1$. Hence, the cofactor matrix can be represented as a 3×3 codeword matrix. For simplicity, we denote by $\bar{z}=\sigma_1(z)$ and $\bar{\bar{z}}=\sigma_1^2(z)$, the conjugates of an entry of the codeword matrix. The cofactor codeword matrix is then defined by

$$\widetilde{\mathbf{Z}} = \begin{bmatrix} z_1 & z_2 & z_3 \\ j\bar{z}_3 & \bar{z}_1 & \bar{z}_2 \\ j\bar{\bar{z}}_2 & j\bar{\bar{z}}_3 & \bar{\bar{z}}_1 \end{bmatrix}, \quad (14)$$

where each diagonal $\widetilde{\mathbf{Z}}_i=\mathbf{M}_1[s_i,s_{i+1},s_{i+2}]^T,\ i=1,\dots,3$. Since $\mathbf{\Gamma}=\mathbf{UZV}$, we denote $\widetilde{\mathbf{\Gamma}}$ its cofactor matrix. It is given by $\widetilde{\mathbf{\Gamma}}=\mathbf{V}^\dagger\widetilde{\mathbf{Z}}\mathbf{U}^\dagger$ and satisfies

$$\mathbf{\Gamma}\widetilde{\mathbf{\Gamma}} = \mathbf{UZVV}^\dagger\widetilde{\mathbf{Z}}\mathbf{U}^\dagger = \det(\mathbf{Z})\mathbf{I}. \quad (15)$$

Developing the cofactor matrix $\widetilde{\mathbf{\Gamma}}$, we prove that it can be written as function of a subspace of a cyclic division algebra of order 9 over the base field $\mathbb{F}=\mathbb{Q}(j)$. This non-commutative algebra is based on the cyclic Galois group $\mathrm{G}=\mathrm{Gal}(\mathbb{K}_1/\mathbb{F})\mathrm{Gal}(\mathbb{K}_2/\mathbb{F})$. Let σ be its generator $\mathrm{Gal}(\mathbb{K}/\mathbb{F})=\langle\sigma\rangle$, we have $\sigma=\sigma_1\sigma_2$ and $\sigma^2\sigma_1=\sigma_2^2$, $\sigma\sigma_1^2=\sigma_2$. From the expression of $\widetilde{\mathbf{\Gamma}}$, we define the elements X_i as

$$X_1 = z_1+\zeta_9\bar{\bar{z}}_2+\zeta_9^2\bar{z}_3\ ;\ X_2=\bar{z}_1+\zeta_9 z_2+\zeta_9^2\bar{\bar{z}}_3$$
$$X_3 = \bar{\bar{z}}_1+\zeta_9\bar{z}_2+\zeta_9^2 z_3 \quad (16)$$

and their conjugates $\sigma^k(X_i)$ by the embeddings $\sigma^k, k=0,\dots,2$ with $\sigma^0(X_i)=X_i$. We have $X_2=\sigma_1(X_1)$ and $X_3=\sigma_1^2(X_1)$ the conjugates of X_1 by the embeddings σ_1^k. Thereby, the cofactor matrix can be rewritten as

$$\widetilde{\mathbf{\Gamma}} = \begin{bmatrix} X_1 & j^2\sigma_1(X_1) & j^2\sigma_1^2(X_1) \\ \sigma_2(X_1) & j^2\sigma(X_1) & j\sigma\sigma_1(X_1) \\ \sigma_2^2(X_1) & j\sigma^2\sigma_1^2(X_1) & j^2\sigma^2(X_1) \end{bmatrix}. \quad (17)$$

Finally, computing the product distance of this matrix, we get the product of all the 9 conjugates of $\tilde{X}\in\mathbb{K}$ and thus

$$d_{\mathrm{p}} = \prod_{i=1}^{9}|\tilde{X}_i| = |N_{\mathbb{K}/\mathbb{F}}(\tilde{X})| = \frac{1}{\sqrt{d_{\mathbb{K}}}} = \frac{1}{\sqrt{49^3 27^3}}. \quad (18)$$

As a result, the elements of $\widetilde{\boldsymbol{\Gamma}}$ are all non-zero unless $\mathbf{s} = \mathbf{0}$ which concludes our proof on the full-diversity of the 3×3 code $\boldsymbol{\Gamma}$, hence its delay-tolerance for arbitrary delay profiles.

D. 4×4 Code based on 4×4 Perfect Code

Similarly to the 2×2 case, the 4×4 code is derived from the tensor product of two algebras over $\mathbb{F} = \mathbb{Q}(i)$. The first one is the cyclic division algebra of degree 4 $\mathcal{A}_1(\mathbb{K}_1/\mathbb{F}, \sigma_1, i)$ used to construct the perfect code in dimension 4. Let $\theta_1 = \zeta_{15} + \zeta_{15}^{-1} = 2\cos(\frac{2\pi}{15})$, the relative extension is $\mathbb{K}_1 = \mathbb{Q}(i, \theta_1)$ of degree $[\mathbb{K}_1 : \mathbb{F}] = 4$ and its relative discriminant is $d_{\mathbb{K}_1} = 1125$. The cyclic Galois group $\mathbb{K}_1/\mathbb{F}$ is generated by $\sigma_1 : \zeta_{15} + \zeta_{15}^{-1} \mapsto \zeta_{15}^2 + \zeta_{15}^{-2}$. The integral basis is given by $\mathrm{B}_1 = \{v_k\}_{k=1}^4 = \{(1-3i)+i\theta_1^2, (1-3i)\theta_1 + i\theta_1^3, -i+(-3+4i)\theta_1 + (1-i)\theta_1^3, (-1+i) - 3\theta_1 + \theta_1^2 + \theta_1^3\}$. The complex rotated lattice of $\mathbb{Z}[i]^4$ is then generated by the unitary matrix

$$\mathbf{M_1} = \frac{1}{\sqrt{15}} \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \\ \sigma_1(v_1) & \sigma_1(v_2) & \sigma_1(v_3) & \sigma_1(v_4) \\ \sigma_1^2(v_1) & \sigma_1^2(v_2) & \sigma_1^2(v_3) & \sigma_1^2(v_4) \\ \sigma_1^3(v_1) & \sigma_1^3(v_2) & \sigma_1^3(v_3) & \sigma_1^3(v_4) \end{bmatrix}. \quad (19)$$

The second relative extension $\mathbb{K}_2$ is chosen such that its degree is 4 over $\mathbb{F}$ and has coprime discriminant with $\mathbb{K}_1$. Let $\mathbb{K}_2 = \mathbb{Q}(\zeta_{16})$ this cyclotomic extension with $d_{\mathbb{K}_2} = 256$ and $\zeta_{16} = e^{i\pi/8}$ the primitive 16^{th} root of unity. The cyclic Galois group is generated by $\sigma_2 : \zeta_{16} \mapsto i\zeta_{16}$. The integral basis of $\mathbb{K}_2$ is $\mathrm{B}_2 = (1, \zeta_{16}, \zeta_{16}^2, \zeta_{16}^3)$ and the lattice generator matrix in $\mathbb{Z}[i]^4$ is given by

$$\mathbf{M_2} = \frac{1}{2} \begin{bmatrix} 1 & \zeta_{16} & \zeta_{16}^2 & \zeta_{16}^3 \\ 1 & i\zeta_{16} & -\zeta_{16}^2 & -i\zeta_{16}^3 \\ 1 & -\zeta_{16} & \zeta_{16}^2 & -\zeta_{16}^3 \\ 1 & -i\zeta_{16} & -\zeta_{16}^2 & i\zeta_{16}^3 \end{bmatrix}. \quad (20)$$

Then, the cross-product algebra defines the compositum of both cyclic extensions as $\mathbb{K} = \mathbb{K}_1\mathbb{K}_2 = \mathbb{Q}(i, 2\cos(\frac{2\pi}{15}), \zeta_{16})$ of order 16 over $\mathbb{Q}(i)$ and accordingly the 16-dimensional complex lattice is generated by the 16×16 unitary matrix $\mathbf{M}_{16} = \mathbf{M}_2 \otimes \mathbf{M}_1$.

The 16 codeword elements are derived from the linear combination $\mathbf{M}_{16}\mathbf{s}$ of q-QAM information symbols. They are then distributed in the 4×4 codeword matrix and assigned the coefficients $\phi_l, l = 1, \ldots, 16$ as

$$\boldsymbol{\Gamma}(\mathbf{s}) = \begin{bmatrix} \phi_1 x_1 & \phi_5 x_5 & \phi_9 x_9 & \phi_{13} x_{13} \\ \phi_2 x_2 & \phi_6 x_6 & \phi_{10} x_{10} & \phi_{14} x_{14} \\ \phi_3 x_3 & \phi_7 x_7 & \phi_{11} x_{11} & \phi_{15} x_{15} \\ \phi_4 x_4 & \phi_8 x_8 & \phi_{12} x_{12} & \phi_{16} x_{16} \end{bmatrix}. \quad (21)$$

1) Non-vanishing minimum determinant: The coefficients ϕ_l are restricted to $|\phi_l| = 1$ and should satisfy the NVD criterion. Therefore, as in previous dimensions, computing the code determinant using symbolic computation under Mathematica, we find that such coefficients corresponding to the Fourier matrix coefficients in $\mathbb{Q}(i)$ allow to get a 4×4 space-time code with the same determinant as the perfect code. We have

$$\Phi = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}. \quad (22)$$

Therefore, the 4×4 infinite code $\boldsymbol{\Gamma}(\mathbf{s})$ has non-vanishing minimum determinant $\delta_{\min}(\mathcal{C}) = 1/d_{\mathbb{K}_1} = 1/1125$, and the 4×4 codeword matrix is defined for $x_1 = X$ by

$$\boldsymbol{\Gamma}(\mathbf{s}) = \begin{bmatrix} X & \sigma_2(X) & \sigma_2^2(X) & \sigma_2^3(X) \\ \sigma_1(X) & i\sigma_2\sigma_1(X) & -\sigma_2^2\sigma_1(X) & -i\sigma_2^3\sigma_1(X) \\ \sigma_1^2(X) & -\sigma_2\sigma_1^2(X) & \sigma_2^2\sigma_1^2(X) & -\sigma_2^3\sigma_1^2(X) \\ \sigma_1^3(X) & -i\sigma_2\sigma_1^3(X) & -\sigma_2^2\sigma_1^3(X) & i\sigma_2^3\sigma_1^3(X) \end{bmatrix} \quad (23)$$

2) Delay-tolerance: In order to examine the delay-tolerance aspect of this code, we enumerate all types of delay profiles. Let a, b, c, d be integer numbers with $a \neq b \neq c \neq d$ and $0 \leq a, b, c, d \leq 3$, we can define four types of profiles as:

- Type 1 of form $\mathfrak{d} = (a, b, c, d)$, *e.g.,* $\mathfrak{d} = (0, 1, 2, 3)$
- Type 2 of form $\mathfrak{d} = (a, a, b, c)$, *e.g.,* $\mathfrak{d} = (2, 2, 1, 0)$
- Type 3 of form $\mathfrak{d} = (a, a, b, b)$, *e.g.,* $\mathfrak{d} = (1, 1, 0, 0)$
- Type 4 of form $\mathfrak{d} = (a, a, a, b)$, *e.g.,* $\mathfrak{d} = (3, 0, 0, 0)$

Each of the asynchronous shifted codeword matrices corresponding to these profiles is full rank if and only if it includes a square 4×4 matrix that is full rank, *i.e.,* a 4×4 minor that is non-zero. Indeed, we have proved this based on some algebraic analysis. Due to lack of space, we only present the main ideas of these proofs.

- For Types 1 and 4, the 4×4 minors can be equal either to the product of some components of the codeword matrix $\boldsymbol{\Gamma}$: $\prod_{1 \leq l \leq 16} |\phi_l x_l|$ or to the product of one component and a 3×3 minor $\mathcal{M}_{3\times3}$: $\phi_l x_l \mathcal{M}_{3\times3}$. By following the same analysis of the 3×3 code, we have by construction that all component products $\subseteq d_{\mathrm{p}} = \prod_{k=1}^{16} |x_k|$ are non-zero. In addition, we verify that all 3×3 minors $\mathcal{M}_{3\times3}$ of $\boldsymbol{\Gamma}(\mathbf{s})$ are non-zero that is equivalent to verify that the 16 entries of the cofactor matrix of $\boldsymbol{\Gamma}$ are non-zero for any $\mathbf{s} \neq \mathbf{0}$.

- For Type 2, we find 4×4 minors that are equal to $\mathcal{M}_{4\times4} = (\phi_j x_j)(\phi_k x_k)\mathcal{M}_{2\times2}$, where $\mathcal{M}_{2\times2}$ are 2×2 minors with components $\phi_l x_l$ such that only one $\phi_l = \pm i$. We prove that these 2×2 minors are non-zero for any $\mathbf{s} \neq \mathbf{0}$, based on the fact that $\mathcal{M}_{3\times3}$ are non-zero according to Type 1.

- For Type 3, we can distinguish two cases. In the first case, $a = 2, 3, b = 0$, there exist 4×4 minors that are equal to the product of two 2×2 minors such that these $\mathcal{M}_{2\times2}$ have their components $\phi_l x_l$ with only one $\phi_l = \pm i$, hence are non-zero according to Type 2. In the second case, $a = 1, b = 0$, the 4×4 minors are functions of 2×2 minors $\mathcal{M}_{2\times2}$ as

$$\mathcal{M}_{4\times4} = \sum_{k=0}^{2} \prod_{l=0}^{1} \mathcal{M}_{2\times2,k+l+1}. \quad (24)$$

Based on more complicated analysis of nested sequences of fields, we prove that this sum is non-zero over $\mathbb{Z}[i]$.

This concludes our proof on the full-rank asynchronous codeword matrices for all types of profiles, hence the delay-tolerance of the 4×4 code $\boldsymbol{\Gamma}$ for any arbitrary delay profile.

IV. Simulation Results

In this section, we evaluate the performance of the proposed 3×3 distributed space-time code used by the relays in synchronous as well as asynchronous transmission compared to its corresponding perfect code. The performance of the 2×2 scheme has already been presented in [14]. Recalling the cooperative system model of Section II, a virtual MIMO scheme is assumed with $M = 3$ transmit antennas and $N_r = 3$ receive antennas. The decoding is performed using the Sphere Decoder. However in the case of asynchronous relays, the codewords are transmitted over $T + \mathfrak{d}_{\max}$ symbol intervals resulting in rank deficiency of the channel matrix. In order to tackle this problem, the MMSE-DFE preprocessing is required to precede the lattice decoding so that the transformed channel has always full rank.

The performance are represented in terms of Codeword Error Rate (CER) versus signal-to-noise ratio E_b/N_0 per receive antenna, which is adjusted as

$$\frac{E_b}{N_0}\Big|_{\text{dB}} = \frac{E_s}{N_0}\Big|_{\text{dB}} - 10 \log R \qquad (25)$$

where E_s is the average energy per receive antenna and R is the code rate in bit per channel use (bpcu).

In the 3×3 schemes, 9 modulated symbols carved from 4-HEX constellation $(\in \mathbb{Z}[j])$ are transmitted at a rate of $R = \frac{18}{3+\mathfrak{d}_{\max}}$ bpcu, where $\mathfrak{d}_{\max} = 0, 2$ is the maximum delay and $\mathfrak{d} = (2, 1, 0)$ the delay profile in asynchronous transmission. In Figure 1, we can observe that both the perfect code and the new code have the same performance for synchronous relays. Whereas for asynchronous relays, the delay-tolerant code preserves the diversity and provides a gain of 5 dB over the 3×3 perfect code at CER of 10^{-4} for $\mathfrak{d}_{\max} = 2$.

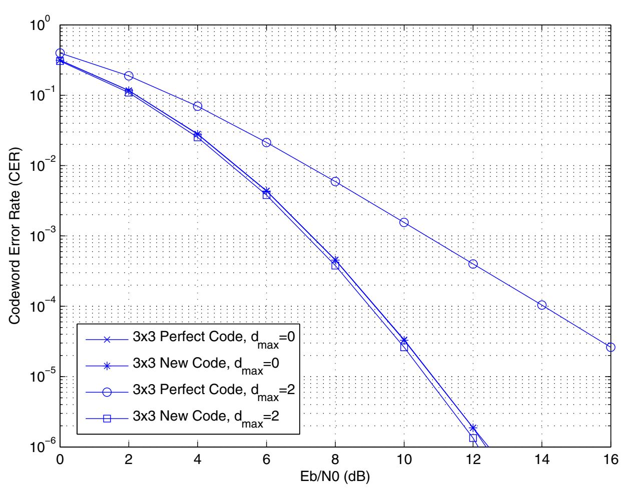

Fig. 1. Performances of 3×3 codes w/wo delay

V. Conclusion

In this paper, we have proposed new delay-tolerant space-time codes based on CDAs of perfect codes. Using tensor product of the perfect code CDA with another CDA of the same order M over the same base field and whose Galois extensions have coprime discriminants, we build rotated lattices in higher dimension in order to construct $M \times M$ codes, for $M = 2, 3, 4$. A key parameter in the construction is the coefficients ϕ_l that allow to preserve the same properties of the perfect codes in synchronous transmission. We have found that ϕ_l corresponding to the coefficients of the Fourier matrix in dimension M yield the same non-vanishing determinants as the perfect codes. These codes besides having full-rate, full-diversity, uniform energy per transmit antennas ($|\phi_l| = 1$) and are information lossless, they have the NVD property and thus are optimal DMT achieving in synchronous case.

In addition, for asynchronous transmission, we have proved that the new codes preserve their full-diversity and are delay-tolerant for arbitrary delay profiles. This property is obtained thanks to the non-zero product distances over $\mathbb{Z}[i]$ or $\mathbb{Z}[j]$ and the full-rank minors of the delayed matrices.

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