Bounded Delay-Tolerant Space Time Codes with Optimal Rates for Two Cooperative Antennas

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***Abstract*—For distributed antennas based communications, the received signal from different sources can be asynchronous due to the propagation or processing delays. This can destroy the space time block code properties designed initially for synchronous case. In this paper, we introduce the delay-diversity tradeoff showing that it is possible to preserve the maximum diversity of a code without reducing its rate as long as the relative delays are in a designed delay tolerance interval. New delay tolerant block codes are proposed and it is shown that they achieve full diversity and optimal rates for both synchronous and asynchronous cases. Their performances are compared to other delay tolerant codes.**

I. Introduction

Recent research works addressed the cooperative communication as a promising technique for future wireless networks. They propose to use distributed antennas as an alternative technique of multiple-antenna (MIMO) systems to provide spatial diversity where network nodes cannot have more than one antenna due to size, cost, or hardware limitations. Moreover, many distributed Space-Time Codes following the well-known rank and determinant criteria [1] were designed to provide specific diversity and coding gain by assuming perfect synchronization among the cooperative nodes or relays. However, due to the distributed nature of the cooperative networks, perfect synchronization is difficult, if not impossible, to achieve. The lack of perfect delay synchronization among the cooperative transmitting nodes destroys the required Space-Time Code signal structure leading to reduction of the achievable diversity.

Several codes preserving these properties in the case of lack of synchronization have been proposed and called “delay tolerant codes”. In [2], the author showed that codes obtained from generalization of the construction in [3] preserve the diversity gain despite the timing offset among the relay nodes. He also showed that certain binary Space-Time Block Codes (STBC) derived from the stacking construction [4] are delay tolerant.

In [5], the authors build on the framework provided by [6] and [7] to design a new class of STBC that shares the advantages of the Threaded Algebraic Space-Time (TAST) codes and are also delay tolerant. Their proposed codes are referred to as “Distributed TAST codes” with length growing exponentially with the number of relays. However, to achieve a full diversity for any delay, the transmission rate is reduced by the repetition of some symbols. This solution has been applied to the optimal synchronous codes such as the Alamouti code [8] and the Golden code [9]. In [10], a new 2×2 delay tolerant code is proposed based on modification of the Golden code by application of convenient unitary matrices. The idea is to combine differently all the symbols to send, for each antenna and each transmission.

Previous works looked for a relevant solution for any delay value and this usually results in a loss of rate. However, in practical wireless systems the delays are generally bounded, which motivates our work. In this paper, we introduce a delay-diversity tradeoff showing that unlike in [5] it is possible to build a space-time block code achieving the full diversity without any rate reduction as long as the relative delays between the received signals from the distributed antennas belong to a defined set of delays. The new design method of the distributed STBC is described. It is based on pair symbol combinations and permutations which guarantees the STBC high rate (no repetition is needed). Depending on these previous operations the bounds of delay tolerance interval centered on zero (i.e. synchronous case) are deduced.

Besides, for asynchronous case, full-rate cannot be achieved because of the channel uses introduced by the delays. By increasing the length of the new designed codes, the width of the delay tolerance interval is expanded, also the rate loss is reduced and optimal rates approaching the full rate can be achieved. We prove analytically that the maximum diversity is achieved for any delay in the tolerance interval. Moreover, the parameters of the proposed codes are optimized for synchronous and asynchronous cases. Some new code examples are shown to outperform the known delay tolerant STBC. To the best of our knowledge no previous works proposed optimal rate, full diversity and bounded delay tolerant codes.

The paper is organized as follows. In Section II, the system model is described. The delay tolerance of some known STBC is discussed in Section III. Section IV introduces the delay-diversity tradeoff. The new delay tolerant codes structure and the proof that they verify the rank and determinant criteria are given in Section V. The code parameters are optimized in Section VI. Some examples of the new codes are given in Section VII and their performances are compared with other known codes. In Section VIII, we give conclusions.

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II. System Model

We consider a wireless system that consists in two transmitters T_1 and T_2 with one antenna each, and a destination D with n_r antennas. The network model is shown in Figure 1.

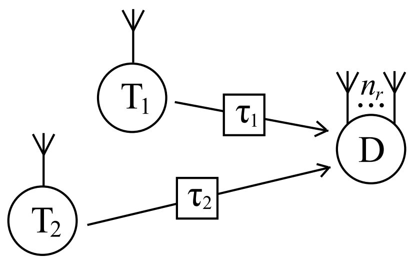

Fig. 1. The asynchronous two-transmitter wireless network model.

Due to the distributed nature of the network, a different time delay is introduced on each transmitter-destination path. τ_1 and τ_2 denote respectively the delays from the transmitters T_1 and T_2 to the destination D, and the relative delay Δ between the two transmitters is equal to: $\Delta = \tau_2 - \tau_1$.

The fractional delays are assumed to be absorbed in multipath (cf. [5]), so the delays τ_1, τ_2 and Δ are integer factors of the symbol period. The delays are unknown at the transmitters, but are known at the destination. The system model is equivalent to a distributed MIMO system with two transmit antennas (one per transmitter) and n_r receive antennas.

Suppose that the two transmitters want to transmit the following N-symbol frame $\mathbf{S} = [S_1, S_2, \dots, S_N]$ to the destination. The signal $\mathbf{y}$ received by the destination is:

$$\mathbf{y} = \mathbf{H}\,\mathbf{S} + \mathbf{n}\,,$$

$\mathbf{n}$ is the Additive White Gaussian Noise (AWGN) at the destination D with variance N_0. The channel is assumed to be quasi-static, so the channel transfer matrix $\mathbf{H}$ is constant over a frame interval but is independent from one frame to another. We denote by $h_{1,i}$ and $h_{2,i}$ the channel gains between T_1 and T_2 respectively and the ith antenna of D with $i = 1, \dots, n_r$.

This system model represents a general cooperative network model with the transmitters being two relay nodes or two base stations or two mobiles able to relay etc...

III. STBC Delay Tolerance

We propose to discuss first some examples of optimal synchronous STBC to introduce the "delay tolerance" notion and next to remind some known solutions that make them delay tolerant. For 2×1 MISO scheme, the Alamouti code [8] is designed and proved to achieve a diversity of two with full data rate as it transmits two symbols in two time intervals:

$$\mathbf{A_s} = \begin{bmatrix} x_1 & -x_2^* \\ x_2 & x_1^* \end{bmatrix},$$

where x_1 and x_2 are the symbols to transmit; x_1^* and x_2^* are the complex conjugate of x_1 and x_2 respectively.

However, this scheme has a full diversity with two perfectly synchronized transmitters and it loses this property when the transmitters are not synchronized. In fact, suppose that the second transmitter T_2 has a delay of one symbol period ($\tau_2 = 1$). The code matrix takes, in this case, the following form:

$$\mathbf{A_a} = \begin{bmatrix} x_1 & -x_2^* & 0 \\ 0 & x_2 & x_1^* \end{bmatrix}.$$

By considering $\mathbf{A_a}^H$ the Hermitian transpose of $\mathbf{A_a}$, we have $\det(\mathbf{A_a}.\mathbf{A_a}^H) = |x_1|^2 .(|x_1|^2 + 2\,|x_2|^2)$ which is equal to zero if only $x_1 = 0$. Thus, the imperfect delay synchronization between the two transmitters destroys the Alamouti structure and makes the destination unable to detect the original signal successfully; so the Alamouti code is not delay tolerant.

The Golden code is an optimal Space-Time code for two transmit and two receive antennas MIMO systems [9]. Its code matrix is:

$$\mathbf{G_s} = \frac{1}{\sqrt{5}} \begin{bmatrix} \alpha(x_1 + \theta x_2) & \alpha(x_3 + \theta x_4) \\ \imath\,\bar{\alpha}\,(x_3 + \bar{\theta} x_4) & \bar{\alpha}\,(x_1 + \bar{\theta} x_2) \end{bmatrix},$$

where $\imath = \sqrt{-1}$, $\theta = \frac{1+\sqrt{5}}{2}$, $\alpha = 1 + \imath(1-\theta)$, $\bar{\theta} = 1 - \theta$, and $\bar{\alpha} = 1 + \imath\,\theta$. However, it is not delay tolerant as can be seen by shifting the second row one column and then setting the entries x_1 and x_2 to zero.

Delay tolerant versions of the Alamouti code and the Golden code were proposed in [5]. The idea consists in repeating the second column of the codes. For instance, the delay tolerant version of the Alamouti code is:

$$\mathbf{A_d} = \begin{bmatrix} x_1 & -x_2^* & -x_2^* \\ x_2 & x_1^* & x_1^* \end{bmatrix}.$$

In the sequel, this new form of Alamouti code will be called the Asynchronous Alamouti (AA).

Also, [5] proposed a new delay tolerant for a variation of the Golden code that will be called hereafter the Asynchronous Golden (AG):

$$\mathbf{C_d} = \frac{1}{\sqrt{2(1+r^2)}} \begin{bmatrix} x_1 + \imath r x_4 & r x_2 + x_3 & r x_2 + x_3 \\ x_2 - r x_3 & \imath r x_1 + x_4 & \imath r x_1 + x_4 \end{bmatrix},$$

where $r = \theta - 1$.

Although these new versions of the Alamouti code and the Golden code are delay tolerant since they achieve the maximum diversity for any shifted version of the code matrix, they suffer from a rate loss due to the repetition.

IV. Delay-Diversity Tradeoff

Suppose that the two transmitters send the same frame $\mathbf{S}$ of length N. For instance, when $\tau_2 > \tau_1$, the following frames will be received at the destination D:

$$\begin{array}{llllllll} T_1: & \mathbf{0}^{\tau_1} & S_1 & \dots & S_i & \dots & S_N & \mathbf{0}^{\Delta} \\ T_2: & \mathbf{0}^{\tau_2} & \dots & S_1 & \dots & S_i & \dots & S_N \end{array}$$

$\mathbf{0}^{\tau}$ denotes an all-zero vector of length τ.

It can be shown by outage probability derivation [11], that this scheme gives a full transmit diversity of two when $\Delta \neq 0$ and of one if $\Delta = 0$. The outage probability results for $n_r = 1$ are illustrated in Figure 2. This means that this scheme is fully delay tolerant except for synchronous case and it has this property by the rule that all versions of one symbol should not arrive at the same time to the destination. However, this scheme is not practical because system delays are centered on zero and hence include the synchronous case.

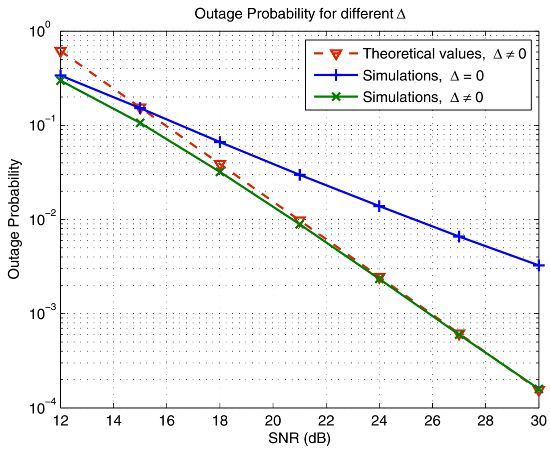

Fig. 2. Outage probability results for $\Delta = 0$ and $\Delta \neq 0$.

To ensure a full diversity when the transmitters are synchronous, we propose to decompose the information frame of length $N > 2$ into two sub-frames named $\mathbf{P_1}$ and $\mathbf{P_2}$ of respective length l_1 and l_2. The first transmitter sends the information frame without changes ($\mathbf{P_1}$ then $\mathbf{P_2}$) and the second transmitter permutes the order of the two parts ($\mathbf{P_2}$ then $\mathbf{P_1}$) and multiply $\mathbf{P_2}$ by a coefficient ϕ (with $|\phi|^2 = 1$) to guarantee a non-zero determinant for the difference of every two distinct scheme frames when $\Delta = 0$. The transmitting scheme becomes:

$$\begin{array}{lcc} T_1: & \mathbf{P_1} & \mathbf{P_2} \\ T_2: & \phi\,\mathbf{P_2} & \mathbf{P_1} \end{array}$$

Using these operations, we can ensure (as shown in next section) a full transmit diversity of two for an interval of relative delays $\Delta \in \{-l_2+1, \ldots, 0, \ldots, l_1-1\}$. This interval will be called the "interval of tolerance".

To have a symmetric interval, we choose the length of the parts $\mathbf{P_1}$ and $\mathbf{P_2}$ to have the closest possible values. Therefore, we choose the following values for the two parts length:

- $l_1 = l_2 = \frac{N}{2}$, when N is an even number,
- $l_1 = [\frac{N}{2}]$, $l_2 = [\frac{N}{2}] + 1$ or the opposite, when N is an odd number; $[x]$ being the integer part of x.

Thus, the interval of tolerance becomes for even N, $\{-\Delta_{\text{max}}, +\Delta_{\text{max}}\}$, with $\Delta_{\text{max}} = \frac{N}{2} - 1$. For odd N, the tolerance interval becomes $\{-\Delta_{\text{max}}, (+\Delta_{\text{max}} - 1)\}$ or $\{(-\Delta_{\text{max}} + 1), +\Delta_{\text{max}}\}$ respectively for $l_1 = [\frac{N}{2}]$ or $l_1 = [\frac{N}{2}] + 1$, with $\Delta_{\text{max}} = [\frac{N}{2}]$.

A relationship exists between the bounds of the interval of tolerance and the length of the frame used to execute the permutation. So, depending on the nature of the network and the maximum relative delay that can exist between the two transmitters, we can expand the width of the interval of tolerance by increasing the frame length. This reflects a trade-off between the tolerated delays and the maximum achievable diversity. In addition, when increasing the length, optimal rates approaching the full rate are achieved for the asynchronous case because the rate loss caused by the additional channel uses induced by the delays will be less important compared to the useful frame length.

V. New Delay Tolerant Codes

The new coding scheme will ensure a full diversity for synchronous and asynchronous distributed networks. It also has a high rate and gives better error rate performance than the other delay tolerant codes introduced in previous sections. The structure of the proposed codes is given in Section V-A. We also prove that the new code family verifies the rank and determinant criteria in Sections V-B.

A. Codes Structure

To increase the rate, we propose to perform a pair combination on an even number N of symbols: $S_1, \ldots, S_N$. The two transmitters combines each consecutive pair of symbols S_{2i-1} and S_{2i} in the following manner:

- $X_i = \mathbf{f}_1\big(\mathbf{a}S_{2i-1} + \mathbf{b}S_{2i}\big)$ for the first transmitter T_1,
- $X'_i = \mathbf{f}_2\big(\mathbf{c}S_{2i-1} + \mathbf{d}S_{2i}\big)$ for the second transmitter T_2,

with $i = 1, \ldots, N'$. The new frame length is $N' = \frac{N}{2}$. $\mathbf{f}_1$ and $\mathbf{f}_2$ are the scaling factors to normalize the power.

Using the permutation strategy presented in Section IV on the new combined symbols, the new codes scheme will have the following form:

$$\mathbf{X} = \begin{bmatrix} X_1 & \ldots & X_{[\frac{N'}{2}]} & X_{[\frac{N'}{2}]+1} & \ldots & X_{N'} \\ \phi X'_{[\frac{N'+1}{2}]+1} & \ldots & \phi X'_{N'} & X'_1 & \ldots & X'_{[\frac{N'+1}{2}]} \end{bmatrix}.$$

By choosing appropriate values for the parameters ($\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$), each pair of points in the constellation of the symbols S_j ($j = 1, \ldots, N$) will be combined into unique symbols X_i and X'_i. In this way, it will be possible to decode the combined symbols due to the bijective mapping. The proposed coding scheme can create a diversity of constellation if the combined symbols X_i and X'_i are different points of a new constellation. For instance, the parameters of the new coding scheme, named the Permutation Code (PC), can have the following values:

- PC1: $\mathbf{a} = 1$, $\mathbf{b} = \theta$, $\mathbf{c} = 1$, $\mathbf{d} = -\theta$ and $\mathbf{f}_1 = \mathbf{f}_2 = \frac{1}{\sqrt{2}}$; where $\theta = e^{\imath\alpha}$ and $\phi = e^{\imath\alpha'}$. α and α' are rotation angles.
- PC2: $\mathbf{a} = \alpha$, $\mathbf{b} = \alpha\theta$, $\mathbf{c} = \bar{\alpha}$, $\mathbf{d} = \bar{\alpha}\bar{\theta}$ and $\mathbf{f}_1 = \mathbf{f}_2 = \frac{1}{\sqrt{5}}$; where α and θ are the coefficient of the Golden code and $\phi = \imath$.

The code matrix for a code length $N' = 6$ has this form:

$$\mathbf{X_6} = \begin{bmatrix} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 \\ \phi X'_4 & \phi X'_5 & \phi X'_6 & X'_1 & X'_2 & X'_3 \end{bmatrix}.$$

B. Rank Criterion

Let $\mathbf{X}$ be the transmitted codeword, and $\mathbf{T}$ be the erroneously decoded codeword at the destination.

Proposition 1: The determinant $\det\big((\mathbf{X}-\mathbf{T}).(\mathbf{X}-\mathbf{T})^H\big)$ *is non-zero for all the values of relative delays* Δ *in the interval* *of tolerance related to the code length N'. Thus, the code matrix has full rank for these values of Δ.*

The proof of Proposition 1 is drawn in Appendix A where different techniques are used depending on the values of Δ (synchronous or asynchronous) and N' (even or odd).

Proposition 2: The pairwise error probability $\mathbb{P}(\mathbf{X} \to \mathbf{T})$ of the new delay tolerant codes for Δ in the interval of tolerance can be upper bounded by:

$$\mathbb{P}(\mathbf{X} \to \mathbf{T}) \leq \left(\prod_{i=1}^{2} \lambda_i\right)^{-n_r} \left(\frac{1}{8N_0}\right)^{-2n_r}. \quad (1)$$

The proof of Proposition 2 is drawn in Appendix B.

In Equation (1), the term $\frac{1}{8N_0}$ represents the Signal to Noise Ratio (SNR) of the network. The diversity order d can be deduced from the PEP as the exponent of the SNR [1]; thus, it is equal to $2.n_r$ for any relative delay Δ in the interval of tolerance.

VI. Code Parameters Optimization

The determinant criterion states that the minimum value of the determinant of matrix $\mathbf{A} = (\mathbf{X} - \mathbf{T}).(\mathbf{X} - \mathbf{T})^H$ over all pairs of different codewords $\mathbf{X}$ and $\mathbf{T}$ should be as large as possible [1]. Here, we optimize the code parameters θ and ϕ that maximize the minimum value of $\det(\mathbf{A})$ derived in Appendix A.

It is important to remind that the parameter ϕ is used to prevent the determinant from becoming equal to zero when $\Delta = 0$ and thus this parameter does not interfere with the code performance when $\Delta \neq 0$.

A. Synchronous Case

For an even code length N', based on the derivation in Appendix A-1, the minimum value of the code determinant is equal to the minimum value of $\det(\mathbf{B}_m.\mathbf{B}_m^H)$ with:

$$\mathbf{B}_m = \begin{bmatrix} X_m & X_{\frac{N'}{2}+m} \\ \phi X'_{\frac{N'}{2}+m} & X'_m \end{bmatrix}.$$

The code parameters that maximizes the value of $\det(\mathbf{B}_m.\mathbf{B}_m^H)$ for M-QAM symbols S are those of the Golden code (PC2).

For an odd code length N', the minimum value of $\det(\mathbf{A})$ is reached when the minimum number of positive terms (4) are not null. The optimized parameters can be obtained by maximizing this minimum determinant.

B. Asynchronous Case

Based on Appendix A-2, for N' even or odd, the minimum value of the code determinant is reached when it contains the minimum number of terms because it is a sum of positive terms; thus the two codewords $\mathbf{X}$ and $\mathbf{T}$ should differ in only one symbol at position m: $e_m = X_m - T_m \neq 0$ and $e'_m = X'_m - T'_m \neq 0$ (with $m \in \{1, N'\}$).

Therefore, the minimum value of the determinant of $\mathbf{A}$ is:

$$\min \det(\mathbf{A}) = |e_m|^2.|e'_m|^2 . \quad (2)$$

The maximization of (2) will define the choice of the optimal code parameters. Here, the configuration of parameters PC1 gives higher values than those of PC2 and we need only to optimize the value of $\theta = e^{\imath\alpha}$. In Figure 3, we plot the values of the minimal determinant (2) in function of α for symbols S_i that belong to 4-QAM and 16-QAM constellations. For 4-QAM constellation, the maximum of value 1 (0.8 with PC2) is reached for values of α in the interval $[30°, 60°] \equiv [\frac{\pi}{6}, \frac{\pi}{3}]$. On the other hand, the maximum of value 0.04 (0.032 with PC2) for 16-QAM constellation is achieved for the following values of α: $30°(\frac{\pi}{6})$, $45°(\frac{\pi}{4})$ and $60°(\frac{\pi}{3})$.

We can also obtain, by mathematical derivation, the same optimal values of α. Indeed, Eq. (2) can be written as $|Z_{2m-1} - \theta^2 Z_{2m}|^2$ with Z_{2m-1} and Z_{2m} being the differences between the S symbols of the two codewords $\mathbf{X}$ and $\mathbf{T}$ at positions $(2m-1)$ and $(2m)$ respectively. To have the optimal α, we calculate the values of θ that maximize $|Z_{2m-1} - \theta^2 Z_{2m}|^2$ for all possible pairs of Z_{2m-1} and Z_{2m}, and then we get the same values of α as in the simulations.

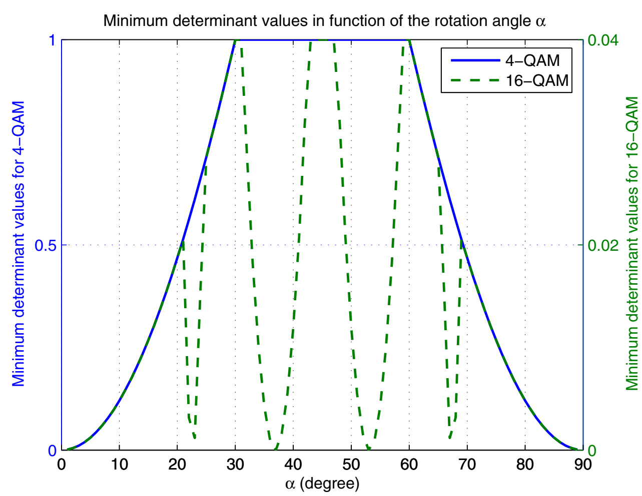

Fig. 3. Minimum determinant values for PC1 parameters with $\Delta \neq 0$.

VII. Code Examples and Numerical Results

Here, we compare the performance of the new delay tolerant Permutation Code (PC) to the Asynchronous Alamouti (AA) for $n_r = 1$ and to the Asynchronous Golden (AG) for $n_r = 2$.

Let us consider the PC for a length code $N' = 6$: $\mathbf{X_6}$.

To compare to the AA, we use the PC1 parameters with $\theta = \phi = e^{\imath\frac{\pi}{2}}$. The symbols S_j sent using the PC belong to the BPSK constellation. To have the same spectral efficiency, the AA sends symbols x_i that belong to the 8-PSK constellation. On the contrary, when compared to the AG, the PC2 parameters are used and the symbols S_j belong to a 4-QAM constellation. AG sends symbols x_i that belong to a 8-PSK constellation.

In Figures 4 and 5, we plot the Frame Error Rate (FER) and the Bit Error Rate (BER) for PC, AA and AG for $\Delta = 0, 1$. For the detection, the exhaustive Maximum Likelihood (ML) is used. From the simulation results, we can notice that the PC designed in this paper outperforms the Asynchronous Alamouti and the Asynchronous Golden when the two transmitters are synchronized or not. This is mainly due to the avoidance of symbol repetition in the PC which increases its rate in comparison to the other codes and thus leads to better error rate performances.

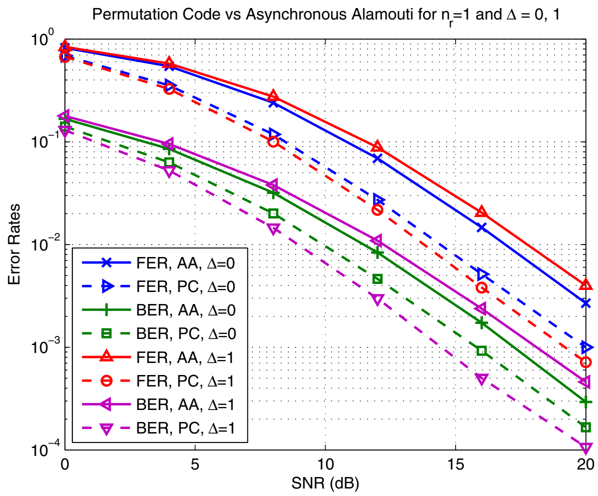

Fig. 4. Error rates comparison between PC and AA for $n_r = 1$.

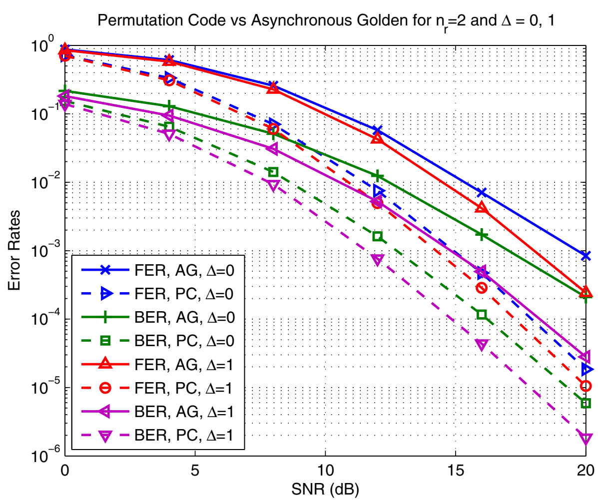

Fig. 5. Error rates comparison between PC and AG for $n_r = 2$.

VIII. Conclusions

In this paper we proposed a new family of delay tolerant codes for two distributed transmitting antennas. These codes are shown to achieve the maximum diversity for a bounded interval of delays including the synchronous case. The code parameters optimization is addressed and solutions are proposed depending on the synchronous and asynchronous cases. The performances are shown to be better compared to some known delay tolerant codes. In future work, we will generalize these design rules to propose new bounded delay tolerant codes for any n_t transmitting antennas.

Appendix A

Without loss of generality, some assumptions are taken to make the presentation of the derivations below clearer:

- We consider that $\tau_2 \geq \tau_1$ and hence the relative delay Δ between the two transmitters is positive: $\Delta = \tau_2 - \tau_1 \geq 0$.
- The destination is considered to be synchronous with transmitter T_1 so that $\tau_1 = 0$.

At the destination, the received signal $\mathbf{y}$ can be written as:

$$\mathbf{y} = \mathbf{H}\,\mathbf{X} + \mathbf{n},$$

where

$$\mathbf{X} = \begin{bmatrix} X_1 & X_2 & \dots & X_{N'} & \mathbf{0}^{\Delta} \\ \mathbf{0}^{\Delta} & \phi X'_{[\frac{N'+1}{2}]+1} & \phi X'_{[\frac{N'+1}{2}]+2} & \dots & X'_{[\frac{N'+1}{2}]} \end{bmatrix};$$

$$\mathbf{H} = \begin{bmatrix} h_{1,1} & h_{2,1} \\ \vdots & \vdots \\ h_{1,n_r} & h_{2,n_r} \end{bmatrix}; \quad \mathbf{n} = \begin{bmatrix} n_{1,1} & n_{2,1} & \dots & n_{(N'+\Delta),1} \\ \vdots & \vdots & \ddots & \vdots \\ n_{1,n_r} & n_{2,n_r} & \dots & n_{(N'+\Delta),n_r} \end{bmatrix};$$

$$\mathbf{y} = \begin{bmatrix} y_{1,1} & y_{2,1} & \dots & y_{(N'+\Delta),1} \\ \vdots & \vdots & \ddots & \vdots \\ y_{1,n_r} & y_{2,n_r} & \dots & y_{(N'+\Delta),n_r} \end{bmatrix}.$$

Let $\mathbf{T}$ be the erroneously decoded codeword at the receiver; thus

$$\mathbf{X}-\mathbf{T} = \begin{bmatrix} e_1 & \dots & e_{[\frac{N'}{2}]} & e_{[\frac{N'}{2}]+1} & \dots & e_{N'} & \mathbf{0}^{\Delta} \\ \mathbf{0}^{\Delta} & \phi\, e'_{[\frac{N'+1}{2}]+1} & \dots & \phi\, e'_{N'} & e'_1 & \dots & e'_{[\frac{N'+1}{2}]} \end{bmatrix}$$

is the codeword difference matrix, with $e_i = X_i - T_i$, $e'_i = X'_i - T'_i$; $\left(i = 1, \dots, N'\right)$.

Let $\mathbf{A}$ be the matrix:

$$\mathbf{A} = (\mathbf{X}-\mathbf{T}).(\mathbf{X}-\mathbf{T})^H = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \tag{3}$$

where:

- $a_{11} = |e_1|^2 + \ldots + |e_{N'}|^2$;
- $a_{12} = \left(\phi^* e'^*_{[\frac{N'+1}{2}]+1} . e_{\Delta+1}\right) + \ldots + \left(e'^*_{[\frac{N'+1}{2}]-\Delta} . e_{N'}\right)$;
- $a_{21} = a_{12}^*$;
- $a_{22} = |e'_1|^2 + \ldots + |e'_{N'}|^2$.

Hereafter, we prove that $\det(\mathbf{A})$ is different from zero for different values of Δ in the interval of tolerance.

1) Synchronous Case ($\Delta = 0$): First, let us consider the case when the code length N' is even. By permuting the columns of the code matrix $\mathbf{X}$, it will take the following form:

$$\mathbf{X} = \begin{bmatrix} X_1 & X_{\frac{N'}{2}+1} & \dots & X_i & X_{\frac{N'}{2}+i} & \dots & X_{\frac{N'}{2}} & X_{N'} \\ \phi X'_{\frac{N'}{2}+1} & X'_1 & \dots & \phi X'_{\frac{N'}{2}+i} & X'_i & \dots & \phi X'_{N'} & X'_{\frac{N'}{2}} \end{bmatrix}.$$

This way, the code matrix will be the concatenation of 2×2 matrices $\mathbf{B}_i = \begin{bmatrix} X_i & X_{\frac{N'}{2}+i} \\ \phi X'_{\frac{N'}{2}+i} & X'_i \end{bmatrix}$, with $(i = 1, \dots, \frac{N'}{2})$.

We have $\det(\mathbf{B}_i) = X_i.X'_i - \phi X_{\frac{N'}{2}+i} X'_{\frac{N'}{2}+i} \neq 0$ except when all the symbols are null, and this is due to ϕ.

Since $\mathbf{B}_i\mathbf{B}_i^H$ are positive definite matrices, we use the determinant inequality in [12]:

$$\det\left(\mathbf{X}\mathbf{X}^H\right) = \det\left(\sum_{i=1}^{\frac{N'}{2}}\left(\mathbf{B}_i\mathbf{B}_i^H\right)\right) \geq \min_{\mathbf{X}\neq\mathbf{0}^{2\times N'}} \sum_{i=1}^{\frac{N'}{2}} \det\left(\mathbf{B}_i\mathbf{B}_i^H\right).$$

By considering the codeword difference matrices, it is easy to deduce the following:

$$\det\left(\mathbf{A}\right) \geq \min \sum_{i=1}^{\frac{N'}{2}} \det\left(\mathbf{D}_i\mathbf{D}_i^H\right); \text{ with } \mathbf{D}_i = \begin{bmatrix} e_i & e_{\frac{N'}{2}+i} \\ \phi\, e'_{\frac{N'}{2}+i} & e'_i \end{bmatrix},$$

$$\text{and } \det\left(\mathbf{D}_i\mathbf{D}_i^H\right) = |e_i|^2\,|e'_i|^2 \;+\; |e_{\frac{N'}{2}+i}|^2\,|e'_{\frac{N'}{2}+i}|^2 - 2\,\mathcal{R}\left(\phi.e_i^*.e_i'^*.e_{\frac{N'}{2}+i}.e'_{\frac{N'}{2}+i}\right) \geq 0.$$

$\mathcal{R}(x)$ being the real part of the complex number x.
Because $\mathbf{X}$ and $\mathbf{T}$ are two different codewords, $\det\left(\mathbf{D}_i\mathbf{D}_i^H\right) \neq 0$ is verified for at least one value of i, and $\det\left(\mathbf{A}\right) = \det\left((\mathbf{X}-\mathbf{T}).(\mathbf{X}-\mathbf{T})^H\right)$ cannot be equal to zero.

On the contrary, when N' is odd, we cannot decompose the code matrix as done above. By calculating the determinant of $\mathbf{A}$ based on Equation (3) and after some suitable grouping, $\det(\mathbf{A})$ can be written as the sum of positive terms having the following form:

$$|e_i|^2\,|e'_j|^2 + |e_k|^2\,|e'_\ell|^2 - 2\,\mathcal{R}\left(p.e_i^*.e_j'^*.e_k.e'_\ell\right), \qquad (4)$$

where i can be equal or different from j but k is different from ℓ, and p can have one of these values $\{1, \phi, \phi^*\}$. The terms (4) cannot be all null at the same time because the codewords $\mathbf{X}$ and $\mathbf{T}$ should differ in at least one symbol, so $\det(\mathbf{A})$ cannot be equal to zero.

2) Asynchronous Case ($\Delta \neq 0$): Here, for N' even or odd, the code matrix cannot be decomposed into submatrices for all the asynchronous values of Δ in the interval of tolerance; so we derive the determinant of matrix $\mathbf{A}$ (Eq. (3)) for these values of Δ. By manipulating the terms of $\det(\mathbf{A})$ conveniently, the determinant can be written as the sum of two groups of terms:

- The terms in the first group are similar to (4).
- The second group terms are products of two squared modulus: $|e_i|^2\,|e'_j|^2$; i can be equal or different from j.

The terms in the two groups above are positive and so $\det(\mathbf{A})$ is a sum of positive terms that cannot be all null at the same time except for the case when the two codewords $\mathbf{X}$ and $\mathbf{T}$ are identical: $e_i = 0$ and $e'_i = 0$ for $i = 1, \ldots, N'$; but this cannot happen because $\mathbf{X}$ and $\mathbf{T}$ should be two different codewords.

Appendix B

Assuming a Maximum Likelihood (ML) detection, the pairwise error probability $\mathbb{P}(\mathbf{X} \to \mathbf{T})$ can be upper bounded by the exponential bound [1]:

$$\mathbb{P}(\mathbf{X} \to \mathbf{T}) \leq \exp\left(-\frac{\mathbb{E}_{\mathbf{H}}\left(\|\mathbf{H}.(\mathbf{X}-\mathbf{T})\|^2\right)}{8N_0}\right), \qquad (5)$$

where $\mathbb{E}_{\mathbf{H}}$ is the expectation over $\mathbf{H}$.

$\|\mathbf{H}.(\mathbf{X}-\mathbf{T})\|^2 = \mathbf{H}.(\mathbf{X}-\mathbf{T}).(\mathbf{X}-\mathbf{T})^H.\mathbf{H}^H = \mathbf{H}.\mathbf{A}.\mathbf{H}^H$.

$\mathbf{A}$ being a Hermitian matrix, we can find a unitary matrix $\mathbf{V}$ and a positive real diagonal matrix $\mathbf{D}$ with $\mathbf{A} = \mathbf{V}^H\mathbf{D}\mathbf{V}$:

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix};$$

where λ_1 and λ_2 are the eigenvalues of matrix $\mathbf{A}$. We shall note that λ_1 and λ_2 are different from zero for relative delays in the interval of tolerance because $\det(\mathbf{A})$ is not null for these values of Δ as seen in Appendix A.
Therefore we have:

$$\mathbb{E}_{\mathbf{H}}\left(\|\mathbf{H}.(\mathbf{X}-\mathbf{T})\|^2\right) = \mathbb{E}_{\mathbf{H}}\left(\mathbf{H}.\mathbf{V}^H.\mathbf{D}.\mathbf{V}.\mathbf{H}^H\right) = \mathbb{E}_{\mathbf{H}}\left(\sum_{i=1}^{2}\sum_{j=1}^{n_r}\lambda_i\,|\beta_{ji}|^2\right) = \sum_{i=1}^{2}\sum_{j=1}^{n_r}\lambda_i\,\mathbb{E}_{\mathbf{H}}\left(|\beta_{ji}|^2\right), \qquad (6)$$

where $\beta_{ji} = \mathbf{h}_j.\mathbf{v}_i$; $\mathbf{h}_j$ being the jth row of $\mathbf{H}$ and $\mathbf{v}_i$ the ith column of $\mathbf{V}$. Because the terms of $\mathbf{v}_i$ constitute a base in $\mathbb{C}^2$ and the $h_{i,j}$ are Gaussian variables with zero mean and a variance of 0.5 for the real dimension; thus, β_{ji} are also Gaussian variables with zero mean and a variance of 0.5 for the real dimension and $\mathbb{E}_{\mathbf{H}}\left(|\beta_{ji}|^2\right) = 1$.

Substituting (6) into (5), we obtain:

$$\mathbb{P}(\mathbf{X} \to \mathbf{T}) \leq \prod_{j=1}^{n_r}\exp\left(-\frac{1}{8N_0}\sum_{i=1}^{2}\lambda_i\right)$$

$$\leq \left(\frac{1}{\prod_{i=1}^{2}\left(1+\frac{\lambda_i}{8N_0}\right)}\right)^{n_r} \leq \left(\prod_{i=1}^{2}\lambda_i\right)^{-n_r}\left(\frac{1}{8N_0}\right)^{-2n_r}.$$

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