This full text paper was peer reviewed at the direction of IEEE Communications Society subject matter experts for publication in the IEEE Globecom 2010 proceedings.

General Construction Method of Bounded Delay-Tolerant Space Time Block Codes

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Abstract—**In distributed antenna networks, the received signal from different transmitters can be asynchronous due to the processing or propagation delays. This destroys the space time code properties designed initially for synchronous case. We introduce, in this paper, a new design method to construct optimal-rate delay-tolerant codes from existing synchronous codes for a certain number of delay profiles that can exist in the network. Some construction examples based on optimal known codes are proposed and it is shown that they achieve full diversity for synchronous and some asynchronous cases. Their performance is compared to other delay tolerant codes.**

I. Introduction

The space-time coding technique has shown to be very useful when multiple antennas exist at the transmitter and/or the receiver because it increases the diversity order and the rate of these systems. In order to give optimal performances, the Space-Time codes (STC) need to follow the well-known rank and determinant criteria [1]. Recently, distributed versions of STC were used in cooperative communications in which the nodes of a network may help each other by relaying their information to the destination. Unlike the multiple-antenna (MIMO) systems where the antennas are collocated at the same device, the antennas in cooperative systems are spatially distributed on different nodes. This new configuration can result in an asynchronism due to the difference in local oscillators and the different propagation delays. The lack of perfect synchronization among the cooperative transmitting nodes destroys the required Space-Time Code signal structure leading to the reduction of the achievable diversity and thus deteriorates the code performance. Therefore, the optimal synchronous STC designed for MIMO systems are no longer valid for asynchronous cooperative communications.

Lately, several codes and design solutions were investigated in the purpose of preserving the properties of STC in the presence of asynchronism. These types of codes are called "delay tolerant". In [2], the author showed that codes obtained from generalization of the construction in [3] preserve the diversity gain despite the timing offset among the relay nodes. He also showed that certain binary Space-Time Block Codes (STBC) derived from the stacking construction [4] are delay tolerant. In [5], the authors built on the framework provided by [6] and [7] to design a new class of space time codes based on the Threaded Algebraic Space-Time (TAST) codes and are also delay tolerant. Their proposed codes are referred to as "Distributed TAST codes" with length growing exponentially with the number of relays. However, to achieve a full diversity for any delay, the transmission rate is reduced by the repetition of some symbols. This solution has been applied to the optimal synchronous codes such as the Alamouti code [8] and the Golden code [9]. In [10], a new 2×2 delay tolerant code is proposed based on modification of the Golden code by application of convenient unitary matrices. The idea is to combine differently all the symbols to send, for each antenna and each transmission.

Previous works looked for a relevant solution for any delay value and this usually results in a loss of rate and an increasing complexity at the receiver. However, in practical wireless systems the delays are generally bounded, which motivates our work. In this paper, we give a new design construction, based on optimal synchronous codes, to build delay tolerant codes for certain delay profiles but without decreasing the rate. The new design method is based on the concatenation of several code matrices and the reordering of the code columns by permutations. The new codes will be referred to as "Bounded Delay Tolerant STBC". A construction example based on the Golden code for two transmitting nodes is considered. It is shown that the new design code achieves the maximal diversity for any delay in a certain tolerance interval. Moreover, the bounded delay tolerant code verifies the rank and determinant criteria and it outperforms some known delay tolerant STBC. For more than two transmitters, TAST codes are used to build new bounded delay tolerant codes.

The paper is organized as follows. In Section II, the system model is described. The delay tolerance of some known STBC and existing solutions are discussed in Section III. The new construction method of Bounded Delay Tolerant STBC is introduced in Section IV. Design examples are given for two transmitters in Section V and for more than two transmitters in Section VI. In Section VII, we give conclusions.

The research leading to these results has received funding from the European Commission's seventh framework program FP7-ICT-2009 under grant agreement n°247223 also referred to as ARTIST4G.

II. System Model

We consider a wireless system with M transmitters $T_1, T_2, \ldots, T_M$ having one antenna each, and a destination D with N antennas. The network model is shown in Figure 1. Due to the distributed nature of the network, a different

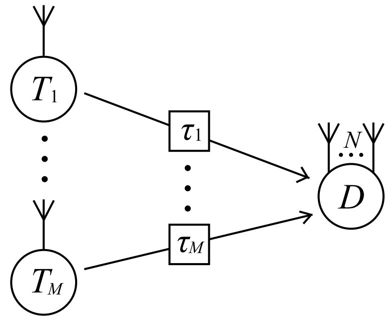

Fig. 1. Asynchronous wireless network with M transmitters.

time delay is introduced on each transmitter-destination path. $\tau_1, \tau_2, \ldots, \tau_M$ denote respectively the delays from the transmitters $T_1, T_2, \ldots, T_M$ to the destination D. We consider, for instance, the first transmitter T_1 as the node reference and we denote by $\Delta_i (i = 2, \ldots, M)$ the relative delay between the transmitter T_i and T_1: $\Delta_i = \tau_i - \tau_1$.
The fractional delays are assumed to be absorbed in multipath (cf. [5]), so the delays τ_i are integer factors of the symbol period. The delays are unknown at the transmitters, but are known at the destination. The system model is equivalent to a distributed MIMO system with M transmit antennas (one per transmitter) and N receive antennas.

The transmission is modeled as follows. The destination D receive the signal: $\mathbf{y} = \mathbf{H}\mathbf{X} + \mathbf{n}$, where $\mathbf{X}$ is the modulated $M \times T$ space-time codeword matrix transmitted over T symbol intervals and $\mathbf{n}$ is the Additive White Gaussian Noise (AWGN) at the destination with variance N_0. The channel is assumed to be quasi-static, so the channel transfer matrix $\mathbf{H}$ is constant over a frame interval but is independent from one frame to another. We denote by $h_{i,j}$ the channel gain between the ith transmitter T_i and the jth antenna of D with $i = 1, \ldots, M$ and $j = 1, \ldots, N$.

This system model represents a general cooperative network model with the transmitters being relay nodes or base stations or mobiles able to relay etc...

In what follows, $(.)^*$, $(.)^t$ and $(.)^H$ denote respectively the conjugate, transpose and conjugate transpose operations.

III. Delay Tolerance of Optimal STBC

We discuss first some examples of optimal synchronous STBC to introduce the "delay tolerance" notion and we remind after some solutions proposed to design delay tolerant codes.

A. Optimal STBC Are Not Delay Tolerant

The Golden code is an optimal STBC for two transmit and two receive antennas MIMO systems [9]. Its code matrix is:

$$\mathbf{G} = \frac{1}{\sqrt{5}} \begin{bmatrix} \alpha(x_1 + \theta x_2) & \alpha(x_3 + \theta x_4) \\ \imath\, \bar{\alpha}\, (x_3 + \bar{\theta} x_4) & \bar{\alpha}\, (x_1 + \bar{\theta} x_2) \end{bmatrix},$$

where $\imath = \sqrt{-1}$, $\theta = \frac{1+\sqrt{5}}{2}$, $\alpha = 1 + \imath(1 - \theta)$, $\bar{\theta} = 1 - \theta$, and $\bar{\alpha} = 1 + \imath\,\theta$. However, it is not delay tolerant as can be seen by shifting the second row one column and then setting the entries x_1 and x_2 to zero. Similarly, the more general class of space-time codes derived from cyclic division algebras (CDA) of which the Golden code is a special case is not delay tolerant either [5].

In [11], the Diagonal Algebraic SpaceTime (DAST) codes were developed for $M \times 1$ MIMO systems based on algebraic number fields. The codewords of the DAST occupy exactly one thread. Subsequently, the authors of [7] showed that, in a $M \times N$ MIMO system, M independent DAST codes (or other constituent codes) could be transmitted simultaneously on different threads with full spatial diversity guaranteed on each thread by tweaking the constituent codes to lie in different algebraic subspaces. The authors named these designs TAST codes and proved that they provide excellent performance and flexibility with respect to signaling constellation, transmission rate, number of transmit and receive antennas, and decoder complexity. Unfortunately, the TAST codes and other related codes available in the literature are not suited for asynchronous cooperative communications, since they are not delay tolerant which is clearly illustrated in [5].

B. Existing Solutions for Delay Tolerant STBC

The CDA and TAST codes are not delay tolerant because they are based on threads of minimal delay $(T = M)$ and hence contain diagonal matrices that are not delay tolerant. In [5], the authors extended the class of the TAST codes to the case of delay tolerant codes for cooperative diversity. Their proposed codes are based on delay tolerant threaded structures of length growing exponentially with the number of relays. The different threads are separated by different algebraic or transcendental numbers which guarantee a nonzero determinant for the difference of every two distinct code words. The idea is to repeat the symbols in a way that, even when the transmitters are asynchronous, the versions of the same symbols sent by the M transmitters arrive at the destination in at least M different symbol periods and thus conserve a full transmit diversity order M. Although these codes provide full-diversity gain for any delay profile, they are not minimum delay length because of symbols repetition and are no longer delay tolerant if one deletes one column of the code word matrix.
This solution was applied to a variant of the Golden code by repeating the second column of the code. For instance, the delay tolerant version of the Golden code, that will be called hereafter the Asynchronous Golden (AG), is:

$$\mathbf{C_d} = \frac{1}{\sqrt{2(1+r^2)}} \begin{bmatrix} x_1 + \imath r x_4 & r x_2 + x_3 & r x_2 + x_3 \\ x_2 - r x_3 & \imath r x_1 + x_4 & \imath r x_1 + x_4 \end{bmatrix}, \tag{1}$$

where $r = \theta - 1$.
Although this new version of the Golden code is delay tolerant since it achieves the maximum diversity for any shifted version of the code matrix, the AG suffers from a rate loss due to the repetition.

Another solution was given in [12] and [10]. The idea in these papers is to combine differently, all the symbols to send, for each antenna and each transmission; this way, even in the presence of a delay, it is sure that each symbol has at least one version that is not arriving at the same time with the other versions of the same symbol sent by the other transmitters to the destination. It was proven that these codes conserve a full rank code matrices in the asynchronous case and yet have a full diversity. A code example that satisfies these design rules is the 2×2 full-rate full-diversity space-time code mentioned in [10]:

$$\mathbf{D} = \begin{bmatrix} ax_1 + bx_2 - cx_3 - dx_4 & -cx_1 - dx_2 - ax_3 - bx_4 \\ bx_1 + ax_2 + dx_3 - cx_4 & -dx_1 + cx_2 - bx_3 + ax_4 \end{bmatrix}, \tag{2}$$

where $a = \frac{1}{\sqrt{(5+\sqrt{5})(2+\sqrt{2})}}$; $b = \frac{1}{\sqrt{(5-\sqrt{5})(2+\sqrt{2})}}$;

$c = \frac{1}{\sqrt{(5+\sqrt{5})(2-\sqrt{2})}}$; $d = \frac{1}{\sqrt{(5-\sqrt{5})(2-\sqrt{2})}}$.

Suppose that the second row of the matrix $\mathbf{D}$ is shifted one column to the right due to a delay by the second transmitter of one symbol period. In this case, we still have two versions of the four symbols arriving to the destination in different time periods. This code will be called the Damen Code (DC).

The problem in combining all the symbols to send is the increasing complexity at the receiver when the number of transmitters becomes bigger ($M > 2$). For instance, for 3×3 codes in [12], each symbol sent by a transmitter is the combination of nine information symbols.

IV. General Construction Method

In this section, we present a novel design to construct bounded delay tolerant STBC from known non-delay tolerant STBC. This solution is based on the concatenation of L code matrices and the permutation of the new matrix columns in a suitable manner to have a new order of the columns. The new designed code ensures a full-diversity for a set of delay profiles that can occur in the network without a rate reduction because no repetition is needed. By applying this method on optimal synchronous codes (i.e. Golden code, TAST), we ensure the optimality of the new codes in the synchronous case and also their high performance in the asynchronous case.

Let us consider a $M \times T$ code having the following matrix:

$$\mathbf{X}^\ell = \begin{bmatrix} X_{11}^\ell & \dots & X_{1T}^\ell \\ \vdots & \ddots & \vdots \\ X_{M1}^\ell & \dots & X_{MT}^\ell \end{bmatrix}.$$

By concatenating $L \geq 2$ different matrices $\mathbf{X}^\ell$ ($\ell = 1, \dots, L$), the new code matrix becomes:

$$\mathbf{X_c} = \begin{bmatrix} X_{11}^1 & \dots & X_{1T}^1 & \dots & X_{11}^L & \dots & X_{1T}^L \\ \vdots & \ddots & \vdots & \dots & \vdots & \ddots & \vdots \\ \underbrace{X_{M1}^1 \;\; \dots \;\; X_{MT}^1}_{\mathbf{X^1}} & & & \dots & \underbrace{X_{M1}^L \;\; \dots \;\; X_{MT}^L}_{\mathbf{X^L}} & & \end{bmatrix}.$$

After, we reorder the columns of $\mathbf{X_c}$ in the following way:

- First, we place in order the first column of the L code matrices from $\mathbf{X^1}$ to $\mathbf{X^L}$.
- Then, we class the remaining columns from the second to the Tth column using the same order as previously.

This operation can be simply done by permutating the columns of matrix $\mathbf{X_c}$. The permutated matrix will have the following form:

$$\mathbf{X_{pc}} = \begin{bmatrix} X_{11}^1 & \dots & X_{11}^L & \dots & X_{1T}^1 & \dots & X_{1T}^L \\ \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots \\ X_{M1}^1 & \dots & X_{M1}^L & \dots & X_{MT}^1 & \dots & X_{MT}^L \end{bmatrix}.$$

Using these operations, we can ensure (as shown in the next sections) a full transmit diversity of M for a set of delay profiles depending on the number of concatenated code matrices L. The idea behind putting several coded symbols together and permutating the code columns is to eventually prevent the versions of the same symbols from arriving to the destination at the same time in the case when a delay occurs in the network.

By increasing L, we extend the number of tolerated delay profiles by the new codes called Bounded Delay Tolerant STBC. Therefore, we can choose the number of the concatenated codes L to cover the existing delay profiles in the network.

Besides, this construction method has many advantages. First, it does not use symbol repetition and thus it does not decrease the code rate. Moreover, the concatenation of code matrices reduces the number of guard intervals needed between every two consecutive code words to ensure that the constructed space-time code is received without any interference from either the next or the previous code words. Consequently, the proposed design method improves the overall rate of the system.

Furthermore, when the communication is synchronous, simple permutations converse to the ones applied at the transmission can be done at the receiver; hence, each initial code matrix can be decoded separately by using the known quasi-optimal decoding algorithms for these STBC (i.e., sphere decoder [13]).

V. Construction Example for $M = 2$

In the case of two transmitters, we apply the new construction method on the optimal Golden code. For clarity reasons, the Golden code matrix is written in the following way:

$$\mathbf{G}^\ell = \begin{bmatrix} X_1^\ell & X_2^\ell \\ X_3^\ell & X_4^\ell \end{bmatrix},$$

where $X_1^\ell = \frac{1}{\sqrt{5}}\,\alpha(x_1 + \theta x_2)$; $X_2^\ell = \frac{1}{\sqrt{5}}\,\alpha(x_3 + \theta x_4)$;

$X_3^\ell = \frac{1}{\sqrt{5}}\,\imath\,\bar{\alpha}\,(x_3 + \bar{\theta} x_4)$; $X_4^\ell = \frac{1}{\sqrt{5}}\,\bar{\alpha}\,(x_1 + \bar{\theta} x_2)$.

A. Code Structure

By concatenating L Golden code matrices $\mathbf{G}^\ell$, we obtain:

$$\mathbf{G_c} = \begin{bmatrix} X_1^1 \; X_2^1 & X_1^2 \; X_2^2 & \dots & X_1^L \; X_2^L \\ \underbrace{X_3^1 \; X_4^1}_{\mathbf{G^1}} & \underbrace{X_3^2 \; X_4^2}_{\mathbf{G^2}} & \dots & \underbrace{X_3^L \; X_4^L}_{\mathbf{G^L}} \end{bmatrix}.$$

After permutating the columns of $\mathbf{G_c}$, the resulting matrix of the new code is:

$$\mathbf{G_{pc}} = \begin{bmatrix} X_1^1 & X_1^2 & \dots & X_1^L & X_2^1 & X_2^2 & \dots & X_2^L \\ X_3^1 & X_3^2 & \dots & X_3^L & X_4^1 & X_4^2 & \dots & X_4^L \end{bmatrix}.$$

A full transmit diversity of two is ensured for an interval of relative delays $\Delta_2 = \tau_2 - \tau_1$:

$$\Delta_2 \in \{-\Delta_{\max}, +\Delta_{\max}\},$$

with $\Delta_{\max} = L-1$. This interval will be called the "interval of tolerance". The bounds of this interval are deduced from the fact that the same coded symbols (i.e. X_2^1 and X_3^1) will not arrive at the same time at the destination unless the first row is shifted L columns to the left or the second row is shifted L columns to the right.

B. Rank and Determinant Criteria

Let $\mathbf{X}$ be the transmitted codeword, and $\mathbf{T}$ be the erroneously decoded codeword at the destination.

Proposition 1: The determinant $\det\big((\mathbf{X}-\mathbf{T}).(\mathbf{X}-\mathbf{T})^H\big)$ *is non-zero for all the values of relative delays* Δ_2 *in the interval of tolerance. Thus, the code matrix has full rank for these values of* Δ_2.
The proof of Proposition 1 is given in Appendix A.

Proposition 2: The pairwise error probability $\mathbb{P}(\mathbf{X} \to \mathbf{T})$ *of the new delay tolerant code for* Δ_2 *in the interval of tolerance can be upper bounded by:*

$$\mathbb{P}(\mathbf{X} \to \mathbf{T}) \le \left(\prod_{i=1}^{2} \lambda_i\right)^{-N} \left(\frac{1}{8N_0}\right)^{-2N}. \qquad (3)$$

The proof of Proposition 2 is drawn in Appendix B.

In Equation (3), the term $\frac{1}{8N_0}$ represents the Signal to Noise Ratio (SNR) of the network. The diversity order d can be deduced from the PEP as the exponent of the SNR [1]; thus, it is equal to $2.N$ for any relative delay Δ_2 in the interval of tolerance.

C. Numerical Results

Here, we compare the performance of the new bounded delay tolerant code that will be referred to as the Permutation Code (PC) with the Asynchronous Golden (AG) (1) and the Damen Code (DC) (2) for two receive antennas at the destination ($N = 2$). The PC with $L = 3$ concatenated Golden code matrices is considered. To have the same spectral efficiency, PC and DC send 4-QAM constellation symbols x_i, and AG sends symbols x_i that belong to 8-PSK constellation.

In Figures 2 and 3, we plot the Frame Error Rate (FER) and the Bit Error Rate (BER) for PC, AG and DC for $\Delta_2 = 0, 1$. For the detection, the exhaustive Maximum Likelihood (ML) is used. From the simulation results, we can notice that the PC designed in this paper outperforms the AG and the DC when the two transmitters are synchronized or not. This is mainly due to the avoidance of symbol repetition used by the AG and because the PC is based on the optimal Golden code which has higher minimal determinant and thus leads to better error rate performances than the DC.

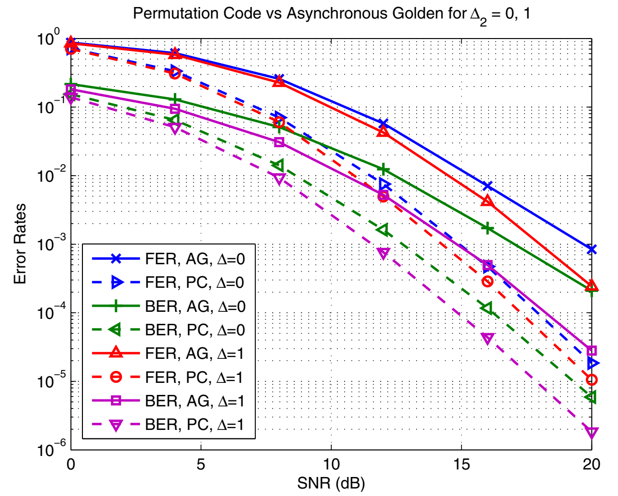

Fig. 2. Error rates comparison between PC and AG.

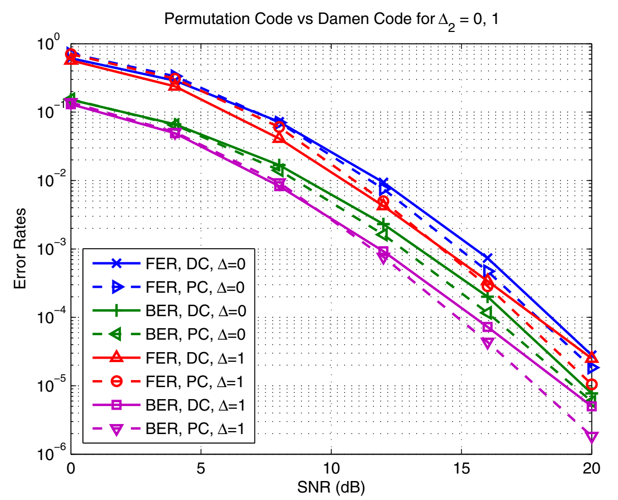

Fig. 3. Error rates comparison between PC and DC.

VI. Construction Example for $M > 2$

For a number of transmitter M bigger than two, an optimal TAST code can be used as the base code. An example of the new design method for three transmitters is given hereafter. The TAST matrix for $M = T = 3$ and $N \ge 3$ receive antennas is [7]:

$$\mathbf{S}^\ell = \begin{bmatrix} s_{11}^\ell & \phi_\ell^{\frac{2}{3}} s_{32}^\ell & \phi_\ell^{\frac{1}{3}} s_{23}^\ell \\ \phi_\ell^{\frac{1}{3}} s_{21}^\ell & s_{12}^\ell & \phi_\ell^{\frac{2}{3}} s_{33}^\ell \\ \phi_\ell^{\frac{2}{3}} s_{31}^\ell & \phi_\ell^{\frac{1}{3}} s_{22}^\ell & s_{13}^\ell \end{bmatrix},$$

where $(s_{j1}^\ell, s_{j2}^\ell, s_{j3}^\ell)^t = \mathbf{M}(x_{j1}, x_{j2}, x_{j3})^t, j = 1, 2, 3$. $\mathbf{M}$ being an optimal 3×3 algebraic rotation matrix and $x_{11}, \dots, x_{33}$ are the information symbols belonging to the constellation used. $\phi_1, \dots, \phi_L$ are appropriate algebraic or transcendental numbers chosen such that the numbers $\{1, \phi_1^{\frac{1}{3}}, \phi_1^{\frac{2}{3}}, \dots, \phi_L^{\frac{1}{3}}, \phi_L^{\frac{2}{3}}\}$ are algebraically independent over the algebraic number field $\mathbb{Q}(\theta)$ that contains the elements of the rotation matrix $\mathbf{M}$ [7].

TABLE I
TOLERABLE RELATIVE DELAY PROFILES FOR $M = T = 3$

Δ_i	Δ_j
0	{-2, -1, 0, 1, 2}
1	{-1, 0, 1, 2}
-1	{-2, -1, 0 ,1}
2	{0, 1, 2}
-2	{-2, -1, 0}
$(\Delta_2, \Delta_3) = (-1, -3); (-2, -4); (3, 1); (4, 2)$	

Let us apply the construction method described in Section IV on $L = 3$ TAST codes $\mathbf{S^1}$, $\mathbf{S^2}$ and $\mathbf{S^3}$. The built code will have the following matrix:

$$\mathbf{S_{pc}} = \begin{bmatrix} s_{11}^1 & s_{11}^2 & s_{11}^3 & \phi_1^{\frac{2}{3}} s_{32}^1 & \dots & \phi_3^{\frac{1}{3}} s_{23}^3 \\ \phi_1^{\frac{1}{3}} s_{21}^1 & \phi_2^{\frac{1}{3}} s_{21}^2 & \phi_3^{\frac{1}{3}} s_{21}^3 & s_{12}^1 & \dots & \phi_3^{\frac{2}{3}} s_{33}^3 \\ \phi_1^{\frac{2}{3}} s_{31}^1 & \phi_2^{\frac{2}{3}} s_{31}^2 & \phi_3^{\frac{2}{3}} s_{31}^3 & \phi_1^{\frac{1}{3}} s_{22}^1 & \dots & s_{13}^3 \end{bmatrix}.$$

The full diversity of $3.N$ is guaranteed for the synchronous case and for a set of relative delay profiles (Δ_2, Δ_3) that depend on each other. Table I provides the different relative delay profiles for which the new code ensures the full diversity: Δ_i corresponds to one of the two relative delays (Δ_2 or Δ_3) and Δ_j to the other one.
The conservation of the full diversity for these set of delay profiles will be possible because the threaded structure (no spatial or temporal interference within a thread) is kept due to the algebraically independent numbers used to separate the threads within a TAST code matrix and with the threads of the other TAST matrices.

VII. CONCLUSIONS

In this paper, a new general design method to construct delay tolerant codes is proposed for M distributed transmit antennas and N receive antennas. We showed that the new codes ensures a full diversity for a set of delay profiles including the synchronous case. For $M = 2$, the new code based on the Golden code verifies the rank and determinant criteria for an interval of relative delays between the two transmitters centered on zero. It also gives better error rate performances than other proposed delay tolerant solutions and codes for synchronous and asynchronous communications. Besides, example based on the TAST codes for $M > 2$ was given with the delay profiles offering a full diversity order.

APPENDIX A

Without loss of generality, some assumptions are taken to make the presentation of the derivations below clearer:

- We consider that $\tau_2 \geq \tau_1$ and hence the relative delay Δ_2 between the two transmitters is positive: $\Delta_2 = \tau_2 - \tau_1 \geq 0$.
- The destination is considered to be synchronous with transmitter T_1 so that $\tau_1 = 0$.

At the destination, the received signal $\mathbf{y}$ can be written as:

$$\mathbf{y} = \mathbf{H}\,\mathbf{X} + \mathbf{n},$$

where

$$\mathbf{y} = \begin{bmatrix} y_{1,1} & y_{2,1} & \cdots & y_{(2L+\Delta_2),1} \\ \vdots & \vdots & \ddots & \vdots \\ y_{1,N} & y_{2,N} & \cdots & y_{(2L+\Delta_2),N} \end{bmatrix};$$

$$\mathbf{H} = \begin{bmatrix} h_{1,1} & h_{2,1} \\ \vdots & \vdots \\ h_{1,N} & h_{2,N} \end{bmatrix};$$

$$\mathbf{X} = \begin{bmatrix} X_1^1 & X_1^2 & \dots & X_2^L & \mathbf{0}^{\Delta_2} \\ \mathbf{0}^{\Delta_2} & X_3^1 & X_3^2 & \dots & X_4^L \end{bmatrix};$$

$$\mathbf{n} = \begin{bmatrix} n_{1,1} & n_{2,1} & \cdots & n_{(2L+\Delta_2),1} \\ \vdots & \vdots & \ddots & \vdots \\ n_{1,N} & n_{2,N} & \cdots & n_{(2L+\Delta_2),N} \end{bmatrix}.$$

$\mathbf{0}^{\Delta_2}$ denotes an all-zero vector of length Δ_2.

Let $\mathbf{T}$ be the erroneously decoded codeword at the receiver; thus

$\mathbf{X} - \mathbf{T} = \begin{bmatrix} e_1^1 & e_1^2 & \dots & e_2^L & \mathbf{0}^{\Delta_2} \\ \mathbf{0}^{\Delta_2} & e_3^1 & e_3^2 & \dots & e_4^L \end{bmatrix}$ is the codeword difference matrix, with $e_i^\ell = X_i^\ell - T_i^\ell$; $\big(i = 1, \dots, 4\big)$ and $\big(\ell = 1, \dots, L\big)$.

Matrix $\mathbf{A}$ will be defined as: $\mathbf{A} = (\mathbf{X} - \mathbf{T}).(\mathbf{X} - \mathbf{T})^H$.
Hereafter, we prove that $\det(\mathbf{A})$ is different from zero for different values of Δ_2 in the interval of tolerance.

1) Synchronous Case ($\Delta_2 = 0$): By making converse permutations to those done to the columns of the code matrix $\mathbf{G_{pc}}$, we obtain the matrix $\mathbf{G_c}$ which is the concatenation of L Golden code matrices $\mathbf{G}^\ell$. It was proven in [9] that $\det(\mathbf{G}^\ell) \neq 0$. Since $\mathbf{G}^\ell {\mathbf{G}^\ell}^H$ are positive definite matrices, we use the determinant inequality in [14]:

$$\begin{aligned} \det\Big(\mathbf{G_c}{\mathbf{G_c}}^H\Big) &= \det\left(\sum_{\ell=1}^{L}\Big(\mathbf{G}^\ell{\mathbf{G}^\ell}^H\Big)\right) \\ &\geq \min_{\mathbf{G_c} \neq \mathbf{0}^{2\times 2L}} \sum_{\ell=1}^{L} \det\Big(\mathbf{G}^\ell{\mathbf{G}^\ell}^H\Big). \quad (4) \end{aligned}$$

By considering the codeword difference matrices, it is easy to deduce from (4) the following:

$$\det\big(\mathbf{A}\big) \geq \min \sum_{\ell=1}^{L} \det\Big(\mathbf{B}^\ell{\mathbf{B}^\ell}^H\Big);\ \text{with } \mathbf{B}^\ell = \begin{bmatrix} e_1^\ell & e_2^\ell \\ e_3^\ell & e_4^\ell \end{bmatrix},$$

$$\text{and } \det\big(\mathbf{B}^\ell{\mathbf{B}^\ell}^H\big) \geq 0. \text{ [9]}$$

Because $\mathbf{X}$ and $\mathbf{T}$ are two different codewords, $\det\big(\mathbf{B}^\ell{\mathbf{B}^\ell}^H\big) \neq 0$ is verified for at least one value of ℓ, and $\det\big(\mathbf{A}\big) = \det\big((\mathbf{X} - \mathbf{T}).(\mathbf{X} - \mathbf{T})^H\big)$ cannot be equal to zero.

2) Asynchronous Case ($\Delta_2 \neq 0$): In the presence of a delay, the code matrix cannot be written as a concatenation of 2×2 Golden code matrices as for the synchronous case. Thus, we derive the determinant of matrix $\mathbf{A}$ for the non-zero relative delays in the interval of tolerance ($0 < \Delta_2 < L$) with the assumptions considered here in the proof).

We have: $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, where:

- $a_{11} = \sum_{i=1}^{2} \Big(\sum_{\ell=1}^{L} |e_i^\ell|^2 \Big)$;
- $a_{12} = \sum_{\ell=1}^{L-\Delta_2} (e_1^{\ell+\Delta_2})(e_3^\ell)^* \; + \; \sum_{\ell=1}^{\Delta_2} (e_2^\ell)(e_3^{L-\Delta_2+\ell})^* \; + \sum_{\Delta_2+1}^{L} (e_2^\ell)(e_4^{\ell-\Delta_2})^*$;
- $a_{21} = a_{12}^*$;
- $a_{22} = \sum_{i=3}^{4} \Big(\sum_{\ell=1}^{L} |e_i^\ell|^2 \Big)$.

By manipulating the terms of $\det(\mathbf{A})$ conveniently, the determinant can be written as the sum of two groups of terms:

- The first group terms are products of squared modulus: $|e_i^\ell|^2 \, |e_{i'}^{\ell'}|^2$ with $i = \{1,2\}$, $i' = \{3,4\}$, ℓ and ℓ' belong to $\{1,\ldots,L\}$.
- The terms in the second group have the following form:
$$|m|^2 + |n|^2 - 2\,\mathcal{R}\big(m^*.\,n\big).$$
$\mathcal{R}\big(x\big)$ being the real part of the complex number x.
m and n are products of e_i^ℓ and $e_{i'}^{\ell'}$.

The terms in the two groups above are positive and so $\det(\mathbf{A})$ is a sum of positive terms that cannot be all null at the same time for two different codewords $\mathbf{X}$ and $\mathbf{T}$.

Appendix B

Assuming a Maximum Likelihood (ML) detection, the pairwise error probability $\mathbb{P}(\mathbf{X} \to \mathbf{T})$ can be upper bounded by the exponential bound [1]:

$$\mathbb{P}(\mathbf{X} \to \mathbf{T}) \leq \exp\left(-\frac{\mathbb{E}_{\mathbf{H}}\left(\|\mathbf{H}.(\mathbf{X}-\mathbf{T})\|^2\right)}{8N_0} \right), \qquad (5)$$

where $\mathbb{E}_{\mathbf{H}}$ is the expectation over $\mathbf{H}$.

$\|\mathbf{H}.(\mathbf{X}-\mathbf{T})\|^2 = \mathbf{H}.(\mathbf{X}-\mathbf{T}).(\mathbf{X}-\mathbf{T})^H.\mathbf{H}^H = \mathbf{H}.\mathbf{A}.\mathbf{H}^H$.

$\mathbf{A}$ being a Hermitian matrix, we can find a unitary matrix $\mathbf{V}$ and a positive real diagonal matrix $\mathbf{D}$ with $\mathbf{A} = \mathbf{V}^H\mathbf{D}\mathbf{V}$:

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix};$$

where λ_1 and λ_2 are the eigenvalues of matrix $\mathbf{A}$. We shall note that λ_1 and λ_2 are different from zero for relative delays in the interval of tolerance because $\det(\mathbf{A})$ is not null for these values of Δ_2 as seen in Appendix A.

Therefore, we have:

$$\begin{aligned} &\mathbb{E}_{\mathbf{H}}\Big(\|\mathbf{H}.(\mathbf{X}-\mathbf{T})\|^2\Big) = \mathbb{E}_{\mathbf{H}}\Big(\mathbf{H}.\mathbf{V}^H.\mathbf{D}.\mathbf{V}.\mathbf{H}^H\Big) = \\ &\mathbb{E}_{\mathbf{H}}\Big(\sum_{i=1}^{2}\sum_{j=1}^{N} \lambda_i \, |\beta_{ji}|^2\Big) = \sum_{i=1}^{2}\sum_{j=1}^{N} \lambda_i \, \mathbb{E}_{\mathbf{H}}\Big(|\beta_{ji}|^2\Big), \end{aligned} \qquad (6)$$

where $\beta_{ji} = \mathbf{h}_j.\mathbf{v}_i$; $\mathbf{h}_j$ being the jth row of $\mathbf{H}$ and $\mathbf{v}_i$ the ith column of $\mathbf{V}$. Because the terms of $\mathbf{v}_i$ form a base in $\mathbb{C}^2$ and the $h_{i,j}$ are Gaussian variables with zero mean and a variance of 0.5 for the real dimension; thus, β_{ji} are also Gaussian variables with zero mean and a variance of 0.5 for the real dimension and $\mathbb{E}_{\mathbf{H}}\left(|\beta_{ji}|^2\right) = 1$.

Substituting (6) into (5), we obtain:

$$\begin{aligned} &\mathbb{P}(\mathbf{X} \to \mathbf{T}) \leq \prod_{j=1}^{N} \exp\left(-\frac{1}{8N_0}\sum_{i=1}^{2}\lambda_i \right) \\ &\leq \left(\frac{1}{\prod_{i=1}^{2}\left(1+\frac{\lambda_i}{8N_0}\right)} \right)^N \leq \left(\prod_{i=1}^{2}\lambda_i\right)^{-N} \left(\frac{1}{8N_0}\right)^{-2N}. \end{aligned}$$

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