

DMT achieving schemes for the Isotropic Fading Vector Broadcast Channel

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Abstract—In this paper, we consider the isotropic fading broadcast channel. This channel refers to the case when no directional information is available at the transmitter side, and was studied by Jafar et al in [1]. It was shown that the isotropic vector broadcast channel (BC-V) can be reduced to an equivalent scalar broadcast channel (BC-S). It is well known from [2]<http://www.pimrc2008.org/> that BC-S is degraded in the same order as the channel magnitude. This implies that the optimal strategy that maximizes the sum capacity for BC-S and BC-V consists on allocating the whole power to the strongest user. Based on these results, we derive in this paper the Diversity Multiplexing Tradeoff (DMT) of the isotropic fading BC-S and BC-V, and we propose optimal schemes that achieve these DMT.

I. INTRODUCTION AND PRIOR ART

Single user MIMO systems have been widely studied in the wireless communications literature because of the significant gain provided in term of capacity over single antenna links. More interestingly, this gain is independent of whether the channel state information (CSI) is available at the transmitter or not.

In the multiuser downlink context with M transmit antennas at the transmitter and K single antenna users, the situation is considerably different. The sum capacity of the broadcast channel (BC) depends largely on the availability of CSI at the transmitter side. When perfect CSI is assumed at both transmitter and receivers, it is well-known that the Dirty Paper Coding (DPC) achieves the maximum sum capacity. Suboptimal linear precoding techniques achieve a large portion of DPC capacity while being simpler to operate than DPC [3]. In both cases, the maximum multiplexing gain that could be achieved is equal to $\min(M, K)$. But the throughput degradation that results from using linear precoding rather than optimal DPC strategies impacts the maximal diversity order [4]. On the other hand, when no CSI is available at the transmitter, the channels of all receivers are statistically identical. Then, the BC channel is degraded in any order and TDMA is the optimal strategy. In this case, the multiuser BC channel is equivalent to a single user MISO channel. The maximal multiplexing gain is one and the maximal diversity gain that can be achieved is M . As we can see, there is a huge gap between the multiuser gains with and without transmit CSI. Since lack of CSI does not lead to multiuser gains and since perfect CSIT is not available, it is interesting to assume

the knowledge of partial CSI at the transmitter.

In this paper, we consider the isotropic fading channel which refers to the case when the transmitter is not able to discriminate between various directions. The transmitter is not able to distinguish between the spatial directions of the users' channels, but is able to track the magnitude of the channel. It has been shown in [1], that under this assumption, the vector broadcast channel (BC-V) is reduced to an equivalent scalar broadcast channel (BC-S). It is well known from [2] that BC-S is degraded in the same order as the channel magnitude. This implies that the optimal strategy that maximizes the sum capacity for both BC-S and BC-V cases consists on allocating the whole power to the user with the strongest channel.

Over a long enough period, the above scheduling strategy maintains also fairness when all users experience the same SNR distribution [5]. It is approximately equivalent to the Proportional Fair Scheduling.

Our objective in this paper is to analyse the Diversity Multiplexing Tradeoff (DMT) $d(r)$ of scalar and vector broadcast channels under a fixed sum rate condition, and to propose optimal coding schemes that achieve this DMT.

The remainder of this paper is organized as follows. We define the system model and the notations used in this paper in section II, and the outage formulation in section III. We derive in section IV the DMT of BC-S and we find the scheme that achieves this DMT. Numerical results are given over a Rayleigh BC-S. In section V, we derive the DMT of BC-V and we find the space time coding scheme that achieves this DMT. Numerical results are also provided over a Rayleigh BC-V. Finally, section VII concludes this paper.

II. SYSTEM MODEL

We consider a K receiver multiple-antenna broadcast channel in which the transmitter has $M \geq 1$ antennas and each receiver has a single antenna. In the following, the scalar broadcast channel BC-S corresponds to $M = 1$, and the vector broadcast channel BC-V corresponds to $M > 1$. The isotropic fading broadcast channel refers to the case when no directional information is available at the transmitter. In other term, the transmitter is only able to track the channel magnitude.

The received signal y_k for user k is given by

$$y_k = \mathbf{h}_k \mathbf{x} + n_k \quad k = 1, \dots, K$$

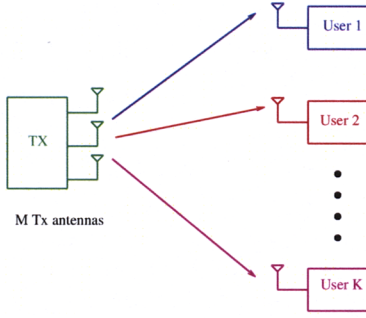


Fig. 1. A Broadcast channel with K users with N antennas each and a single transmitter with M antennas

where $\mathbf{h}_1, \mathbf{h}_2, \dots, \mathbf{h}_K$ are the channel vectors (with $\mathbf{h}_i \in \mathbb{C}^{1 \times M}$) of user 1 to K, with i.i.d unit variance Gaussian entries. The vector $\mathbf{x} \in \mathbb{C}^{1 \times M}$ is the transmitted signal, and n_1, \dots, n_K are independent complex Gaussian noise terms with unit variance. The input must satisfy a transmit power constraint of ρ , i.e., $\mathbb{E}[\|\mathbf{x}\|^2] \leq \rho$. We assume that each receiver has full knowledge of its own channel.

III. OUTAGE FORMULATION

Before going to the DMT analysis, we will start by defining the outage probability

$$P_{\text{out}}(R) = \text{Prob}\{R_{\text{sum}} \leq R\} \doteq \text{SNR}^{-d} \quad (1)$$

d is the diversity gain of the equivalent channel. R corresponds to the maximal sum capacity such that $R_{\text{sum}} = \sum_{i=1}^K R_i$ where R_i denotes the rate assigned to a user i in order to maximize the sum capacity. For the DMT analysis, we let R scales as $r \log \text{SNR}$ with r is the total multiplexing gain.

IV. DMT ANALYSIS FOR THE SCALAR BROADCAST CHANNEL

The results of the DMT analysis are summarized in the following proposition:

Proposition 1: The DMT of an isotropic scalar broadcast channel, with one antenna at the transmitter and K single antenna receivers is given by

$$d_{\text{scalar}}^* = K(1 - r) \quad (2)$$

The optimal scheme achieving this DMT corresponds to the transmission of QAM symbols to the user with the strongest channel.

Based on the results of [2] summarized in section IV-A, the proof of this proposition is given in subsections IV-B and IV-C.

A. Scheduling strategy achieving maximum rate

It is well-known from [2] that the BC-S is degraded in the same order as the channel magnitude. The optimal strategy consists therefore in allocating the power to the strongest user, i.e the user with the best channel. In this case, the maximal sum capacity for the BC-S is given by

$$C_{\text{BC}}(h_1, \dots, h_K, \rho) = \log \left(1 + \rho \max_{i=1 \dots K} |h_i|^2 \right)$$

B. Optimal tradeoff for the scalar broadcast channel

At each time symbol, the user with best channel is selected. The received data at the selected user can be written as

$$y = hx + z \quad (3)$$

where h is the fading of the equivalent channel, such that

$$|h|^2 = \max(|h_1|^2, |h_2|^2, \dots, |h_K|^2)$$

h_i denotes the fading coefficient between the transmitter and the user i . $u_i = |h_i|^2$ is exponentially distributed, its pdf can be expressed as

$$f_{u_i}(u_i) = e^{-u_i}$$

and its cdf is given by

$$F_{u_i}(u_i) = 1 - e^{-u_i}$$

Thus, the pdf of $u = |h|^2$ is given by

$$p(u) = K f_{u_i}(u) (F_{u_i}(u))^{K-1} = K e^{-u} (1 - e^{-u})^{K-1}$$

Let ϵ be an arbitrary small positive number. By approximating e^{-u} by $1 - u$ for small values of u , we get

$$\text{Prob}\{|h|^2 \leq \epsilon\} = \epsilon^K \quad (4)$$

The outage probability for the equivalent SISO case in equation (3) is given by

$$\begin{aligned} P_{\text{out}}(R) &= \text{Prob}\left\{\log_2(1 + |h|^2 \text{SNR}) \leq R\right\} \\ &= \text{Prob}\left\{|h|^2 \leq \frac{2^R - 1}{\text{SNR}}\right\} \end{aligned}$$

By applying the equation (4) to the expression of outage probability at high SNR, we get

$$P_{\text{out}}(R) \approx \left(\frac{2^R - 1}{\text{SNR}}\right)^K$$

The outage probability decays as $1/\text{SNR}^K$, which means that the diversity order of the scalar broadcast channel is K. With $R = r \log_2 \text{SNR}$, the outage probability is given by

$$P_{\text{out}}(R) \doteq \frac{1}{\text{SNR}^{K(1-r)}} \quad (5)$$

The diversity multiplexing tradeoff for the scalar BC is thus:

$$d_{\text{scalar}}^* = K(1 - r) \quad (6)$$

C. The QAM is DMT achieving

For the SISO case, the expression of the symbol error probability [6] for a fixed channel is given by

$$\text{Pe}_h^{\text{QAM}} = 4Q\left(\sqrt{\frac{6 \text{SNR}}{2^R - 1}} |h|^2\right)$$

At high SNR, by replacing R with $r \log_2 \text{SNR}$ and averaging over all the $u = |h|^2$

$$\text{Pe}^{\text{QAM}} = 4 \int_0^\infty Q\left(\sqrt{6 \text{SNR}^{1-r} u}\right) p(u) du$$

After simple manipulations described in appendix A, we get

$$P_e \leq \frac{4K!}{\prod_{i=0}^{K-1} (3 \text{SNR}^{1-r} + (i+1))} \doteq \text{SNR}^{-K(1-r)} \quad (7)$$

The selection of the strongest user in a SISO Broadcast channel achieves the maximum multiuser diversity gain, *i.e.* K , and the full multiplexing gain which is equal to 1. From the expressions of the outage probability in eq (5) and the error probability in eq (7), we can see that the QAM is DMT achieving since the exponents of SNR in their expressions are the same. This is due also from the fact that the QAM is universal over all fading SISO channels, regardless from the distribution of the channel.

V. DMT ANALYSIS FOR VECTOR BROADCAST CHANNEL

The results of the DMT analysis are summarized in the following proposition:

Proposition 2: The DMT of an isotropic vector broadcast channel, with M antennas at the transmitter and K single antenna receivers is given by

$$d_{\text{vector}}^* = MK(1-r) \quad (8)$$

In this case, the one-layered perfect code combined with the selection of the strongest user achieves this DMT.

Based on the result of [1] summarized in section V-A, the proof of this proposition is given in subsections V-B and V-C.

A. Scheduling strategy achieving maximum rate

Lemma 1 (Same as Theorem 6 and Lemma 4 in [1]):

The capacity region of the isotropic vector fading broadcast channel with perfect CSI at the receivers and only a knowledge of magnitudes at the transmitter is identical to the capacity region of the equivalent scalar fading Gaussian broadcast channel. The fading vector broadcast channel is therefore degraded in the same order's as the users' channel magnitudes. It is also well known that the sum capacity for a degraded channel is achieved by single-user transmission to the most capable user. That's why, the optimal strategy in the BC-V case that achieves the sum capacity consists in allocating the whole power to the strongest user.

B. Optimal tradeoff for the vector broadcast channel

The outage probability of the equivalent system is given by

$$\begin{aligned} P_{\text{out}}(R) &= \text{Prob} \left\{ \log_2 \left(1 + \frac{\text{SNR}}{M} \|h\|^2 \right) \leq R \right\} \\ &= \text{Prob} \left\{ \|h\|^2 \leq \frac{M(2^R - 1)}{\text{SNR}} \right\} \end{aligned}$$

where

$$\|h\|^2 = \max \left(\|h_1\|^2, \|h_2\|^2, \dots, \|h_K\|^2 \right) \quad (9)$$

Lemma 2: Let ϵ be an arbitrary small positive number, and $\|h\|^2$ denotes the random variable described by (9), then

$$\text{Prob}(\|h\|^2 \leq \epsilon) \approx \frac{1}{M!} \epsilon^{MK} \quad (10)$$

Proof: See appendix B. ■

At high SNR, the outage probability is obtained by applying lemma 2:

$$P_{\text{out}}(R) \approx \frac{1}{M!} \left(\frac{M(2^R - 1)}{\text{SNR}} \right)^{MK}$$

The outage probability decays as $1/\text{SNR}^{MK}$, which means that the diversity order of the channel that can be achieved using this strategy of scheduling is MK . With $R = r \log_2 \text{SNR}$, the outage probability is given by

$$P_{\text{out}}(r) \approx \frac{M^{MK}}{M!} \left(\frac{\text{SNR}^r}{\text{SNR}} \right)^{MK} = \frac{M^{MK}}{M! \text{SNR}^{MK(1-r)}} \quad (11)$$

The diversity multiplexing tradeoff for the BC-V is thus:

$$d_{\text{vector}}^* = MK(1-r) \quad (12)$$

C. Achieving DMT using one layer of the perfect codes

The DMT of the isotropic BC-V can be achieved using one layer of the perfect code. The received signal is

$$\mathbf{y}_{1 \times M} = \mathbf{h}_{1 \times M} \mathbf{X}_{M \times M} + \mathbf{n}_{1 \times M} \quad (13)$$

where

$$\mathbf{X} = \text{diag}(\mathbf{G}\mathbf{s}) = \begin{pmatrix} x & 0 & \dots & 0 \\ 0 & \sigma(x) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \sigma^{M-1}(x) \end{pmatrix} \quad (14)$$

with $\mathbf{G}\mathbf{G}^\dagger = \mathbf{I}$ and $\mathbb{E}[\mathbf{X}\mathbf{X}^\dagger] = \mathbf{I}$. We assume here that the channel of each user remains constant over one codeword.

The one-layer perfect code converts the MISO channel into M -parallel SISO channels. In order to prove the optimality of these codes, we will prove in a first time that these codes are universal over the equivalent M -parallel SISO channels. This conversion into parallel channels is approximately universal for the class of MISO channels with i.i.d fading coefficients [6], which occurs when the selection strategy is used.

1) Universality of the code: The one-layer perfect code converts the MISO channel into M parallel SISO channels. The equivalent model is given by

$$\mathbf{Y}_{M \times M} = \mathbf{H}_{M \times M} \mathbf{X}_{M \times M} + \mathbf{N}_{M \times M} \quad (15)$$

with $\mathbf{Y} = \text{diag}(\mathbf{y})$, $\mathbf{H} = \text{diag}(\mathbf{h})$ and $\mathbf{N} = \text{diag}(\mathbf{n})$.

Over these M parallel SISO channels, the coding scheme X is universal since X is 1-NVD scheme over the $M \times M$ MIMO channels [Section 1.4.3 in [7]].

Although this conversion is DMT achieving, there is a loss in SNR for the same error probability performance due to the loss in term of capacity.

2) *Outage of the equivalent model:* For the equivalent model in eq (15), the instantaneous capacity per channel use is

$$\begin{aligned} C_{Ts} &= \frac{1}{M} \log \det(\mathbf{I} + \text{SNR} \mathbf{H} \mathbb{E}[\mathbf{X}\mathbf{X}^\dagger] \mathbf{H}^\dagger) \\ &= \frac{1}{M} \log \det(\mathbf{I} + \text{SNR} \mathbf{H}\mathbf{H}^\dagger) \end{aligned}$$

By using [Theorem 16.8.4 in [8]], for any positive definite matrix $M \times M$ matrix \mathbf{A}

$$\det(\mathbf{A}) \leq \left(\frac{\text{Tr}(\mathbf{A})}{M} \right)^M$$

Notice that $\text{Tr}(\mathbf{I} + \text{SNR} \mathbf{H}\mathbf{H}^\dagger) = M + \text{SNR} \|h\|^2$. Then, after simple simplification

$$C_{Ts} \leq \log \left(1 + \frac{\text{SNR}}{M} \|h\|^2 \right) = C_{\text{MISO}} \quad (16)$$

The outage probability of the equivalent system is therefore

$$P_{\text{out},c} = \text{Prob} \{ C_{Ts} \leq r \log \text{SNR} \} \geq P_{\text{out}} \doteq \text{SNR}^{-MK(1-r)}$$

which ends the proof.

VI. NUMERICAL RESULTS

For illustration, we consider an isotropic BC channel with 2 antennas at the transmitter ($M = 2$) and 2 single antenna receivers ($K = 2$). The optimal DMT that can be achieved in this configuration is $4(1-r)$. As shown in section V, the selection of the strongest user combined with the use of one-layer of the Golden code [9] achieves this DMT. The codeword of the strongest user is given by

$$\mathbf{X} = \text{diag} \left(\mathbf{G} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \right)$$

with $\mathbf{G} = \frac{1}{\sqrt{5}} \begin{pmatrix} \alpha & \alpha\bar{\theta} \\ \bar{\alpha} & \bar{\alpha}\theta \end{pmatrix}$. with $\theta = \frac{1+\sqrt{5}}{2}$, $\bar{\theta} = \frac{1-\sqrt{5}}{2}$, $\alpha = 1+i-i\theta$ and $\bar{\alpha} = 1+i-i\bar{\theta}$. s_1 and s_2 denote the information QAM symbols,

Since the one-layer of the perfect code converts the 2×1 MISO channel into 2 parallel SISO channels, we compare these two schemes in terms of capacity and outage behavior. There is a small gap between the capacity of the 2 schemes, as shown in fig. 2. This difference is explained by the relation between their capacities, as presented in eq (16).

This gap corresponds to a 1.5dB loss in SNR in terms of outage probability, as illustrated in fig. 3. Both schemes achieve the same diversity order, that is equal to four. This explains the large gain obtained over the time sharing case, with a diversity order of two.

We also study the performance of the one-layered perfect code in terms of frame error rate. The perfect code for two transmit antennas is the Golden Code presented in [9]. In addition, for this antenna configuration, and only for this one, we can use another universal space-time code, which is the Alamouti one (refer to [6] for more details). The simulation results in fig. 4 show again a difference of 1.5dB between the Alamouti scheme and the one-layered Golden Code, as

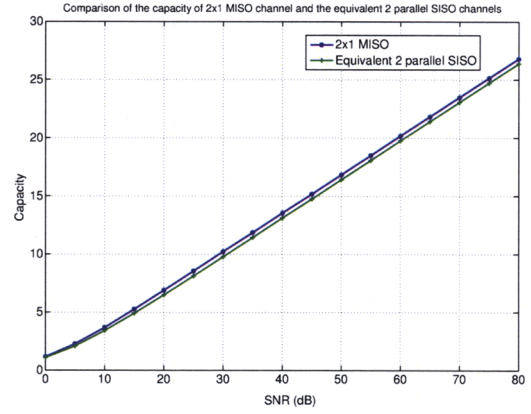


Fig. 2. Comparison of the capacity of the 2×1 MISO channel with the equivalent 2 parallel SISO channels

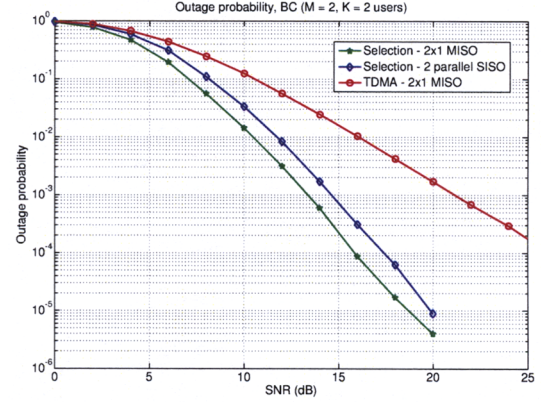


Fig. 3. Comparison of the outage probabilities of the selection with 2×1 MISO channel / the equivalent 2 parallel SISO channels and the time sharing strategy for 2 bits spectral efficiency

expected from the outage behavior. These results also confirm the same level of diversity for both schemes.

In the general case, when the number of transmit antennas is greater than two, there is no equivalence to the Alamouti scheme. The one-layered perfect code is then optimal.

VII. CONCLUSION AND PERSPECTIVE

In this paper, we derived the DMT of the isotropic scalar and vector broadcast channels (BC-S and BC-V), which have been previously studied in [1] from a capacity point of view. It has been shown that the optimal strategy that maximizes the sum capacity for both BC-S and BC-V cases consists in allocating the whole power to the strongest user. For the scalar broadcast channel, the maximal diversity that can be achieved is K , and the full multiplexing gain is equal to one. The optimal coding scheme achieving this DMT is the QAM constellation. For the vector case, the maximal diversity is MK , and the multiplexing gain is equal to one. This loss in

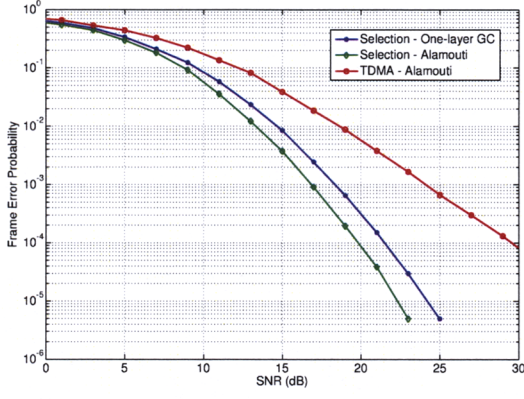


Fig. 4. Comparison of the frame error probabilities of the selection with alamouti / one-layer of the golden code (GC) and the time sharing strategy with alamouti for 2 bits spectral efficiency (QPSK)

degree of freedom is due to the lack of directional information at the transmitter side. Thus, in order to more benefit from multiple transmit antennas, some information about channel directions should be available at the transmitter.

To ensure fairness, a long enough period must be considered. To relax this constraint, a selection algorithm could be proposed taking into account buffer state information and channel state information. This point will be investigated in a future work.

VIII. ACKNOWLEDGMENTS

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APPENDIX A

ERROR PROBABILITY DERIVATION

By using the $Q(u) \leq \exp(-u^2/2)$, the symbol error probability is

$$\text{Pe}^{\text{QAM}} \leq 4 \int_0^\infty \exp(-3 \text{SNR}^{1-r} u) p(u) du$$

Let $\alpha = 3 \text{SNR}^{1-r}$. By expanding the above expression, we get

$$\text{Pe}^{\text{QAM}} \leq 4K \sum_{i=0}^{K-1} \left[\frac{(-1)^i \binom{K-1}{i}}{\alpha + i + 1} \right] = \frac{P(\alpha)}{Q(\alpha)}$$

Notice that,

$$(-1)^i 4K \binom{K-1}{i} = \text{Res} \left(\frac{P(\alpha)}{Q(\alpha)}, -(i+1) \right)$$

Then

$$P[\alpha = -(i+1)] = 4K! \quad \forall i$$

Consequently,

$$\text{Pe} \leq \frac{4K!}{\prod_{i=0}^{K-1} (3 \text{SNR}^{1-r} + (i+1))} \doteq \text{SNR}^{-K(1-r)} \quad (17)$$

APPENDIX B PROOF OF LEMMA 2

$u_i = \|h_i\|^2$ are χ -square distributed with $2M$ degrees of freedom, where the pdf is given by

$$f_{\|h_i\|^2}(u_i) = \frac{1}{(M-1)!} u_i^{M-1} e^{-u_i}$$

and the cdf is

$$F_{\|h_i\|^2}(u_i) = 1 - e^{-u_i} \sum_{m=0}^{M-1} \frac{u_i^m}{m!}$$

Thus, the pdf of $u = \|h\|^2$ is

$$p_{\|h\|^2}(u) = \frac{K}{(M-1)!} u^{M-1} e^{-u} \left[1 - e^{-u} \sum_{m=0}^{M-1} \frac{u^m}{m!} \right]^{K-1}$$

Let ϵ be an arbitrary small value, then

$$\text{Prob}(\|h\|^2 \leq \epsilon) = \int_0^\epsilon p_{\|h\|^2}(u) du = \left(1 - e^{-\epsilon} \sum_{m=0}^{M-1} \frac{\epsilon^m}{m!} \right)^K$$

Notice that

$$\begin{aligned} e^{-\epsilon} \sum_{m=0}^{M-1} \frac{\epsilon^m}{m!} &\approx e^{-\epsilon} \left(e^\epsilon - \frac{1}{M!} \epsilon^M + o(\epsilon^M) \right) \\ &\approx 1 - \frac{1}{M!} \epsilon^M e^{-\epsilon} + o(\epsilon^M) \\ &\approx 1 - \frac{1}{M!} \epsilon^M + o(\epsilon^M) \end{aligned}$$

Then, $\text{Prob}(\|h\|^2 \leq \epsilon) \approx \frac{1}{M!} \epsilon^{MK}$

REFERENCES

- [1] S. A. Jafar and A. Goldsmith, "Isotropic fading vector broadcast channels: The scalar upperbound and loss in degrees of freedom," *IEEE Transactions on Information Theory*, vol. 51, no. 3, pp. 848–857, March 2005.
- [2] L. Li and A. Goldsmith, "Capacity and optimal resource allocation for fading broadcast channels-part I: Ergodic capacity," *IEEE Transactions on Information Theory*, vol. 47, no. 3, pp. 1083–1102, Mar 2001.
- [3] C. B. Peel, B. M. Hochwald, and A. L. Swindlehurst, "A Vector Perturbation Technique for Near Capacity Multiantenna Multiuser Communication Part I: Channel Inversion and Regularization," *IEEE Transactions on Information Theory*, vol. 53, no. 195–202, pp. 4787–4792, January 2005.
- [4] L. Mroueh, S. Rouquette-Léveil, G. R.-B. Othman, and J.-C. Belfiore, "DMT of weighted parallel channels: Application to the broadcast channel," in *Proceedings of ISIT 2008*.
- [5] D. Gesbert and M.-S. Alouini, "How much feedback is multi-user diversity really worth?" in *International Conference on Communications, Paris, France*, June 2004, pp. 20–24.
- [6] D. Tse and P. Viswanath, *Fundamentals of wireless communications*. Cambridge University Press, 2005.
- [7] Y. Sheng, "Cooperative diversity in mimo channels with amplify and forward," Ph.D. dissertation, ENST France, 2007.
- [8] T. M. Cover and J. A. Thomas, *Elements of Information Theory*. New York : Wiley, 1991.
- [9] J.-C. Belfiore, G. Rekaya, and E. Viterbo, "The golden code: A 2×2 full rate space time code with non vanishing determinants," *IEEE Transactions on information theory*, vol. 51, no. 2, pp. 1432–1436, apr 2005.