Reduced-Complexity Lattice Spherical Decoding

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***Abstract*—Lattice sequential decoders based on a spherical search region, such as the Sphere Decoder and the SB-Stack decoder, implement a tree-search strategy to find the ML solution while visiting only the lattice points that belong to a sphere of a predefined radius. Their computational complexity depends then critically on the choice of the initial sphere radius. We propose in this work novel initial sphere radius selection methods for spherical-region based sequential lattice decoders and show through simulations the complexity reduction allowed by such methods when the Sphere Decoder is used while maintaining ML performance.**

***Index Terms*—MIMO systems, Sphere Decoding, Radius selection.**

I. Introduction

Last years have witnessed spectacular developments of wireless networks that have widely transformed all aspects of our daily life. Driven by the emergence of new real-time high-throughput multimedia applications and the success of digital technologies, several network solutions are available today and are thoroughly used in all modes of communications. Main examples include cellular networks, wireless ad-hoc networks and wireless sensor networks involving single or multiple users and/or antennas. In order to ensure a reliable QoS in such communication systems, it is of fundamental importance to develop efficient physical layer technologies. For instance, Multiple-Input Multiple-Output technologies have been used as potential candidates to provide high-rate and reliable transmission in emergent wireless communication systems. A main challenge is such systems is the design and implementation of robust and low-complexity detection algorithms at the receiver devices.

Optimal decoding performance are obtained using ML criterion. An ML detector estimates the nearest vector to the received channel output in the sense of the minimisation of the Euclidean distance. Implementing the ML criterion using an exhaustive search provides optimal performance, nevertheless it requires a high complexity that increases as the size of the constellation codebook or the number of the transmit antennas gets higher. This drawback makes such implementation impossible in practical systems.

In order to reduce the decoding complexity of ML detection, a lattice spherical decoder termed the Sphere Decoder (SD) has been proposed in literature [1, 2]. The idea behind this method is to reduce the number of the visited lattice points in the decoding process to a finite set defined by a spherical region. The algorithm seeks the ML solution inside a sphere of a predefined radius centered at the received point. Limiting the search space to the spherical region allows to reduce the number of visited points and therefore to significantly decrease the overall decoding complexity [3].

The same principle of spherical region used in the SD has been later on adopted in the so called Spherical-Bound Stack decoder (SB-Stack). This algorithm, proposed in [4] combines the best-first tree-search strategy of the Stack decoder [5, 6] with the search region of the Sphere Decoder. While the Stack decoder requires a higher complexity than the SD, the SB-Stack implementing additionally a spherical search region achieves same ML performance as the SD with a reduced complexity of at least 30%.

A main issue of such decoding schemes is the selection of the initial sphere radius. Indeed, the initial radius must be big enough to enclose at least one lattice point. If the latter is too large, there will be too many lattice points in the sphere which results in a high complexity; and if the radius is too small, there will be no points in the sphere. A judicious choice of the radius needs to be made to greatly speed up the decoder. Several methods to the choice of the initial radius have been proposed in literature. The optimal method consists in considering the covering radius of the lattice generated by the channel matrix as a starting radius. Nevertheless, this selection method requires the knowledge of the underlying lattice which is very complex and not obvious to calculate. Authors in [1, 2, 7] propose alternatively to use an upper bound of the covering radius when the latter is not provided. In addition, according to [1], an initial sphere radius can be selected taking into account the statistical characteristics of the noise and authors proposed in [8, 9] a selection method considering the Euclidean distance from the received signal to the ZF estimate.

In this work, we propose novel methods to select the initial value of the sphere radius applicable to spherical lattice decoders such as the Sphere Decoder and the SB-Stack decoder implementable to any linear communication system presenting a lattice representation of the channel output mainly to MIMO channels presented in this work.

The remainder of this paper is organized as follows. In section II, we will expose the system model and review the main sphere-based decoding algorithms. In section III, we will review the existing methods for initial radius selection and present our strategy. We will provide simulation results in section IV and a conclusion in section V.

II. System model and problem statement

Without loss of generality, we consider an $n_t \times n_r$ MIMO system with n_t transmit and n_r receive antennas using spatial

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multiplexing. The complex-valued representation of the channel output is given by:

$$\mathbf{y}_c = \mathbf{H}_c \mathbf{s}_c + \mathbf{w}_c \quad (1)$$

where $\mathbf{H}_c \in \mathbb{C}^{n_r \times n_t}$ denotes the channel matrix of elements drawn i.i.d. according to the distribution $\mathcal{N}(0,1)$ and assumed perfectly known at the receiver. $\mathbf{w}_c \in \mathbb{C}^{n_r}$ is the additive white Gaussian noise of variance $\sigma^2 \mathbf{I}_{n_r}$ and $\mathbf{s}_c$ is composed of the complex-valued information symbols. In order to obtain a lattice representation of the channel output, we apply the complex-to-real transformation to get the real-valued system given by:

$$\begin{bmatrix} \Re(\mathbf{y}_c) \\ \Im(\mathbf{y}_c) \end{bmatrix} = \begin{bmatrix} \Re(\mathbf{H}_c) & -\Im(\mathbf{H}_c) \\ \Im(\mathbf{H}_c) & \Re(\mathbf{H}_c) \end{bmatrix} \begin{bmatrix} \Re(\mathbf{s}_c) \\ \Im(\mathbf{s}_c) \end{bmatrix} + \begin{bmatrix} \Re(\mathbf{w}_c) \\ \Im(\mathbf{w}_c) \end{bmatrix}$$

where $\Re(.)$ and $\Im(.)$ denote respectively the real and imaginary parts of a complex-valued vector. The equivalent channel output can then be written as:

$$\mathbf{y} = \mathbf{H}\mathbf{s} + \mathbf{w} \quad (2)$$

This system is to be considered in the decoding process. When a length-T Space-Time code is used, the channel output can be written in the same form of (1) with the equivalent channel matrix $\mathbf{H}_{eq}$ given by:

$$\mathbf{H}_{eq} = \mathbf{H}_c \mathbf{\Phi} \quad (3)$$

where $\mathbf{\Phi} \in \mathbb{C}^{n_t T \times n_t T}$ corresponds to the coding matrix of the underlying code [3]. For ease of presentation and given that both uncoded and coded schemes result in a same real-valued lattice representation, we consider in the remaining of this work the spatial multiplexing and symmetric case with $n_t = n_r$ and let $n = 2n_t$.

A. ML detection

The objective of the receiver is to obtain an estimate $\hat{\mathbf{s}}$ of the symbols vector $\mathbf{s}$, given $\mathbf{H}$ and $\mathbf{y}$. Under such conditions and assuming coherent systems where $\mathbf{H}$ is perfectly known at the receiver, the optimal detector that minimizes the average error probability is the maximum-likelihood (ML) decoder given by the minimization problem [10]:

$$\hat{\mathbf{s}} = \underset{\mathbf{s} \in \mathcal{A}}{\operatorname{argmin}} \parallel \mathbf{y} - \mathbf{H}\ \mathbf{s} \parallel^2 \quad (4)$$

where $\mathcal{A} \subset \mathbb{Z}$ represents the integer-valued alphabet to which belong the real and imaginary parts of the complex information symbols. Thus the ML detector chooses the message $\mathbf{s}$ which yields the smallest distance between the received vector $\mathbf{y}$, and hypothesized message $\mathbf{Hs}$. The ML detector represents a discrete optimization problem over candidate vectors $\mathbf{s}$ within the chosen constellation. Unfortunately, in the case of higher constellations and higher dimension of the system (number of antennas), the search for the ML solution in an exhaustive way requires a very high complexity.

Alternatively, lattice sequential decoders have been proposed. In literature, there exist mainly two search strategies. The first, introduced by Pohst [11], consists of searching the closest point within a sphere with fixed radius. The second method was proposed by Kannan [12] and takes into consideration a search within a parallelotope. Since the area covered by a parallelotope is larger than that covered by a sphere, which results in higher complexity, the first method was the most studied and used.

B. Spherical lattice decoders

We are interested in this work in spherical lattice decoders, in particular to the selection of the initial sphere radius for such methods. These decoders, such as the Sphere Decoder and the SB-Stack decoder, use a branch and bound principle and their constraint is related to limiting the search region to a spherical one. In [13], authors proposed a mathematical analysis of the SD. A practical implementation of this search strategy was proposed by Viterbo and Boutros [1] and applied for MIMO decoding for fading channels. This algorithm consists of searching the closest point within a sphere around the received point.

Limiting the search region to a sphere imposes a search interval for each detected symbol. To briefly describe this implication mathematically, we need first to move to the tree structure of MIMO systems. To do so, we first perform a pre-decoding step via a QR decomposition of the channel matrix such that $\mathbf{H} = \mathbf{QR}$ where $\mathbf{Q} \in \mathbb{R}^{n \times n}$ is an orthogonal matrix and $\mathbf{R} \in \mathbb{R}^{n \times n}$ is upper triangular. Given the orthogonality of $\mathbf{Q}$, (2) is equivalent to:

$$\tilde{\mathbf{y}} = \mathbf{Q}^t \mathbf{y} = \mathbf{R}\mathbf{s} + \mathbf{Q}^t \mathbf{w} \quad (5)$$

And the decoding problem remains to solve the equivalent system given by:

$$\hat{\mathbf{s}} = \underset{\mathbf{s} \in \mathcal{A}}{\operatorname{argmin}} \|\tilde{\mathbf{y}} - \mathbf{R}\mathbf{s}\|^2 \leq C^2 \quad (6)$$

where C denotes the radius of the sphere inside which is searched the ML solution. Notice that the tree representation obtained through the predecoding phase can be generated by performing Cholesky decomposition of the Gram matrix $\mathbf{G} = \mathbf{H}^t \mathbf{H}$.

Then, to solve the minimization problem, we consider $\rho = \mathbf{R}^{-1}\tilde{\mathbf{y}} = \mathbf{H}^{-1}\mathbf{y}$ and $\xi = \rho - \mathbf{s}$. Since $\mathbf{R}\xi = \mathbf{R}(\rho - \mathbf{s}) = \mathbf{R}(\mathbf{R}^{-1}\tilde{\mathbf{y}}) - \mathbf{R}\mathbf{s} = \tilde{\mathbf{y}} - \mathbf{R}\mathbf{s}$, we can rewrite the inequality $\parallel \tilde{\mathbf{y}} - \mathbf{R}\ \mathbf{s} \parallel^2 \leq C^2$ as:

$$\parallel \mathbf{R}\ \xi \parallel^2 \leq C^2 \quad (7)$$

By developing the necessary computation [1], we obtain search interval $I_i = [b_{inf,i}, b_{sup,i}]$ for each symbol s_i such that:

$$b_{inf,i} = \left\lceil -\sqrt{\frac{T_i}{p_{ii}}} + S_i \right\rceil ; b_{sup,i} = \left\lfloor \sqrt{\frac{T_i}{p_{ii}}} + S_i \right\rfloor$$

where

$$p_{ii} = R_{ii}^2 \quad , p_{ij} = \frac{R_{ij}}{R_{ii}} \quad , i = 1, ..., n, j = i+1, ..., n$$

$$S_i = \rho_i + \sum_{j=i+1}^{n} p_{ij}\xi_j , T_i = C^2 - \sum_{l=i+1}^{n} p_{ll} \left(\xi_i + \sum_{j=l+1}^{n} p_{lj}\xi_j \right)^2$$

The number of visited nodes during the tree-search phase depends then on the interval I_i for each s_i which highly depends on the initial sphere radius C. For a given channel matrix $\mathbf{H}$, the computational decoding complexity of the sphere decoder is given by [1, 2]:

$$O\left(n^2 \times \left(1 + \frac{n-1}{4dC^2}\right)^{4dC^2}\right) \tag{8}$$

where d^{-1} denotes a lower bound for the eigenvalues of the Gram matrix $\mathbf{G} = \mathbf{H}^t\mathbf{H}$.

The same principle of the spherical-bounds search of the SD has been later on adopted in the SB-Stack decoder [4]. This algorithm uses a best-first tree-search strategy like the traditional stack decoder [5] combined with the spherical search region of the SD as defined previously.

It is clear therefore that the choice of the initial radius greatly impacts the decoding complexity of spherical region-based lattice decoders.

III. Initial sphere radius selection

In order to be sure to find a lattice point inside the sphere, the initial radius must be big enough to enclose at least one lattice point. If C is too large, there will be too many lattice points in the sphere; if C is too small, there will be no points in the sphere. A judicious choice of C can greatly speed up the decoder. This is the focus of this work. We address in the following the main existing radius selection strategies as well as our proposed methods.

A. Optimal radius: covering radius

The optimal radius for spherical decoding is the covering radius of the lattice generated by the matrix $\mathbf{H}$. It represents the smallest distance of a lattice point from a deep hole and guarantees to have at least one point inside the sphere. Nevertheless, the computation of this radius requires too much computations and high complexity. Alternatively, an upper bound of the covering radius, known as Roger's bound, can be considered and given by [14]:

$$C_{ub} = \left((n\log n + n\log(\log n) + 5n)\,\frac{\sqrt{\det(\mathbf{H}\mathbf{H}^t)}}{V_n}\right)^{1/n} \tag{9}$$

where V_n represents the volume of the $n-$dimensional sphere of radius 1.

B. Noise statistics-based radius selection

Given the difficulty and the complexity of deriving the covering radius for a given channel matrix, other strategies exist in literature. A first practical way is to choose C adjusted according to the noise variance σ^2 so that the probability of a decoding failure is negligible [8]. In [8], Hassibi and Vikalo have used statistical properties of the noise to propose a radius as a linear function of the noise variance. Thus, we may choose the radius to be scaled to the variance of the noise as [8]:

$$C_1 = 2n\sigma^2 \tag{10}$$

A major drawback of this method is that at low SNR range, the noise variance is quite high resulting in a big sphere radius.

C. Channel fading-based radius selection

In practical implementations, as proposed in [15], it is important to take into consideration, in addition to the noise variance, the fact that fadings affect the lattice form. Indeed, if we consider the case of multidimensional lattices where dimensions are not affected in the same manner, and then point repartition will not be the same and dimensions will not have the same thikness in terms of lattice points. Thus, if this additional constraint is not taken into consideration, we can result in the case where few points are inside the sphere. The solution to this problem consists of choosing a radius C_2 which satisfies:

$$C_2 = \min\left(diag(\mathbf{H}^t\mathbf{H})\right) \tag{11}$$

Finally, the sphere radius using which both channel fadings and noise statistics are taken into consideration can be chosen as:

$$C_0 = \min\left(\left(2n\sigma^2, \min\left(diag\left(\mathbf{H}^t\mathbf{H}\right)\right)\right)\right) \tag{12}$$

Although the existing methods guarantee to have at least a lattice point inside the sphere, they require a high complexity. In this work we propose novel ways to select the initial sphere radius, providing lower complexity than the existing ones.

D. Proposed radius selection method

The idea behind our proposed method is to consider a novel radius as a function of the Euclidean distance from the received signal to the Zero Fording-Decision Feedback Equalizer (ZF-DFE) estimate. Let d_{DFE} denote the euclidean distance from the received signal to the ZF-DFE solution $\mathbf{s}_{\text{DFE}}$ such that:

$$d_{\text{DFE}} = \| \mathbf{y} - \mathbf{H}\mathbf{s}_{\text{DFE}} \| \tag{13}$$

Then, we consider the radius given by:

$$C_3 = f\left(d_{\text{DFE}}\right) \tag{14}$$

where f is a predefined function. We provide in the following two examples of the function f.

1) Example of a multiplicative function: An example of the function used to compute C_3 is

$$f_1\left(d_{\text{DFE}}\right) = \frac{d_{\text{DFE}}}{\sqrt{\alpha}} \tag{15}$$

where $\alpha \in \mathbb{R}^+$ is a real-valued strictly positive parameter. It is clear that for $\alpha = 1$, the initial radius coincids with the Euclidean distance from the received signal to the ZF-DFE solution. To validate the choice of this value of C_3, we carried out several computer simulations and derived the statistics for having the radius $C_3 < C_0$ for different values of the parameter α using the Sphere Decoder. An example corresponding to a 2×2 MIMO system using spatial multiplexing and 16-QAM modulations is plotted through figure.1 in which we evaluate the probability of having $C_3 < C_0$ over 10^5 channel realizations. Numerical results show that for low and moderate

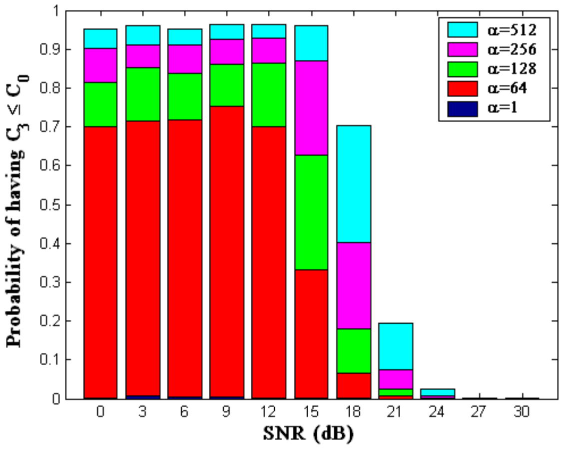

Fig. 1. Probability of having $C_3 \leq C_0$ for the SD in a 2×2 MIMO system using spatial multiplexing.

SNR values, the probability that the radius $C_3 < C_0$ is larger than 50%. It is more judicious in these cases to consider as initial radius C_3 rather than C_0, given, more interestingly, that C_3 suffices to find the ML solution. However, in high SNR regime, the radius C_0 has a smaller value than C_3 with very high probability (almost 99%). In the cases where $C_3 < C_0$, choosing C_3 as initial radius rather than C_0 allows to speed up the algorithm's convergence by visiting less nodes inside the sphere of radius C_3 during the tree search phase of the ML solution. Consequently, in these high probable scenarii, considering C_3 as a starting radius allows to find the ML solution with lower complexity than that if C_0 was used.

The parameter α can be chosen arbitrarily. However, in order to guarantee that the resulting radius C_3 be smaller than C_0, it can be easily proven that this parameter should satisfy $\alpha \geq \alpha_{min}$ where α_{min} is given by:

$$\alpha_{min} = \max\left(\frac{d_{\text{DFE}}^2}{4n^2\sigma^4}, \frac{d_{\text{DFE}}^2}{\min^2\left(diag\left(\mathbf{H}^t\mathbf{H}\right)\right)}\right)$$

2) Example of an additive function: A second example of the positive-valued function f proposed in this work is as follows:

$$f_2\left(d_{\text{DFE}}\right) = d_{\text{DFE}} - \alpha \qquad (16)$$

For this example too, the parameter α can be selected arbitrarily such that the resulting function is positive. In order to have a resulting radius taking into consideration the noise variance as well as the channel fadings, we derived a lower bound α_{min} for the parameter α for this example and obtained:

$$\alpha_{min} = \max\left(d_{\text{DFE}} - 2n\sigma^2, d_{\text{DFE}} - \min\left(diag\left(\mathbf{H}^t\mathbf{H}\right)\right)\right)$$

IV. Performance Evaluation

Before we present numerical results, we compare analytically in the following the total computational complexity in terms of the number of multiplicative operations corresponding to the SD using C_0 as initial radius and SD using the proposed radius C_3.

A. Complexity of SD using C_0

The total computational complexity of the sphere decoder, for a given sphere radius C_0, is counted according to the different steps performed during the decoding process as follows:

- **Pre-decoding:** as mentioned previously, this phase can be performed using QR decomposition of the matrix $\mathbf{H}$ or Cholesky decomposition on the Gram matrix $\mathbf{G} = \mathbf{H}^t\mathbf{H}$ requiring respectively a computational complexity of $\frac{2}{3}n^3$ and $\frac{7}{6}n^3 + n$. As the Gram matrix will be used in the following step for radius calculation, it is more judicious to consider it also during this phase for complexity reduction.
- **Radius calculation:** as the matrix $\mathbf{G}$ has been already computed in the previous step, the radius calculation adds only 2 multiplicative operations to compute the radius C_0.
- **Tree-search:** the complexity of this phase is given according to the previous section by: $O\left(n^2 \times \left(1 + \frac{n-1}{4dC_0^2}\right)^{4dC_0^2}\right)$.

1) Complexity of SD using C_3: Complexity of the SD using the proposed radius selection method is composed of 3 main parts as follows:

1) **Pre-decoding step:** as QR decomposition is needed to find the ZF-DFE solution, we consider it in the pre-decoding step rather than the Cholesky decomposition. Then the computational complexity of this phase is $\frac{2}{3}n^3$.
2) **Initial radius selection:** in this step, $\frac{n(n-1)}{2}$ operations are needed to find the ZF-DFE solution and $n^2 + 1$ multiplicative operations are required to compute the function f, in total, $\frac{3}{2}n^2 - \frac{n}{2} + 1$ operations are needed to compute the radius C_3.
3) **Tree search:** using the SD and for a given radius C_3, the tree search phase computational complexity is given by: $O\left(n^2 \times \left(1 + \frac{n-1}{4dC_3^2}\right)^{4dC_3^2}\right)$.

As discussed above, the proposed method is more interesting than the existing one using the radius C_0 for two reasons: first, its pre-decoding and radius calculation computational complexity are lower, second, when $C_3 < C_0$, which is highly probable, the complexity corresponding to the tree search phase is also reduced which results in an overall alleviated decoding complexity.

B. Numerical results

We present now simulation results evaluating the total complexity (pre-decoding, radius calculation and tree-search phases) for the 2×2 and 4×4 MIMO cases using spatial multiplexing and 16-QAM considering the SD with the commonly used radius C_0 as well as using C_3 are provided. We studied the case of the multiplicative function for different values of α.

Starting with the case of $n_t = n_r = 2$ depicted in figure.2, it can be seen from numerical results that using the proposed radius provides a reduced complexity particularly in the low

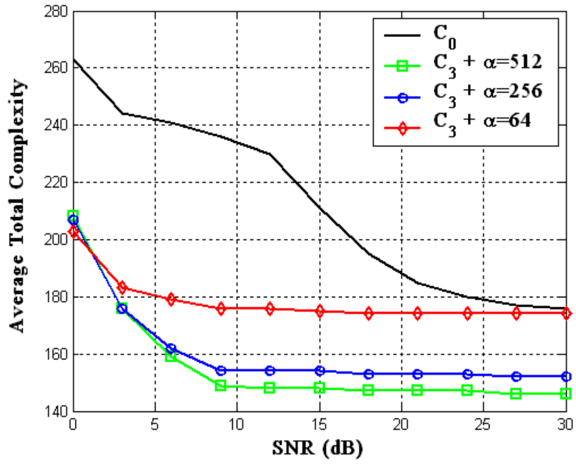

Fig. 2. Average complexity for $n_t = n_r = 2$ using 16-QAM and multiplicative radius function.

and moderate SNR range compared to the case when the initial radius C_0 is used. Notice also that larger values of α, resulting in smaller value of C_3, allow to have lower complexity while guaranteeing no decoding failure, i.e. using this radius, the algorithm is able to find the ML solution inside the underlying sphere. Thus, the complexity reduction is more considerable when the parameter α increases. For example, when $\alpha = 512$, the complexity gain offered using the radius C_3 over that using C_0 is in average at least 20%. Same conclusions can be derived for the case of $n_t = n_r = 4$ plotted in 3. As we can notice, the proposed radius allows to find the ML solution while reducing the total decoding complexity in average by at least 20% compared to the case where the radius C_0 is considered.

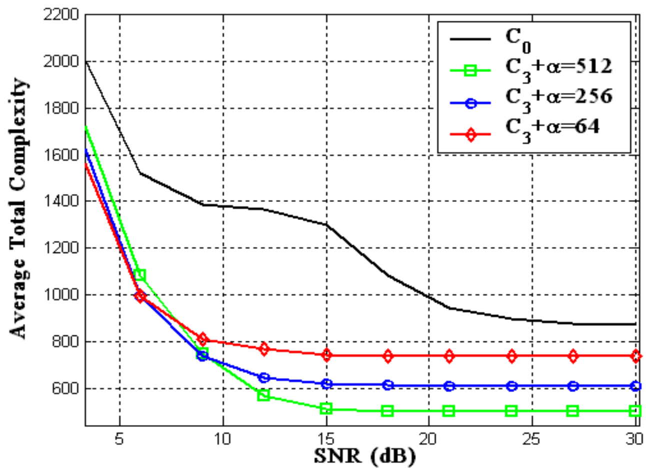

Fig. 3. Average complexity for $n_t = n_r = 4$ using 16-QAM and multiplicative radius function.

V. Conclusion

In this work, novel selection methods for the initial sphere radius value in application to spherical lattice decoders are proposed. The idea behind the proposed techniques is to use the Euclidean distance from the received signal to the ZF-DFE estimate when computing the initial sphere radius. The proposed methods are applicable for example to the Sphere Decoder and the SB-Stack decoder. When applied to the SD, the proposed methods provide a complexity gain of at least 20% over the Sphere decoder implementing the existing radius selection technique.

References

[1] E. Viterbo and J. Boutros. A universal lattice code decoder for fading channels. *IEEE Transactions on Information Theory*, 45(5):1639–1642, July 1999.

[2] O. Damen, A. Chkeif, and J-C. Belfiore. Lattice code decoder for space-time codes. *IEEE Communications Letters*, 4(5):161–163, May 2000.

[3] G. Rekaya and J.-C Belfiore. On the complexity of ml lattice decoders for decoding linear full rate space-time codes. In *IEEE International Symposium on Information Theory*, pages 206–206, June 2003.

[4] G.R. Ben-Othman, R. Ouertani, and A. Salah. The spherical bound stack decoder. In *Proceedings of International Conference on Wireless and Mobile Computing*, pages 322–327, October 2008.

[5] F. Jelinek. Fast sequential decoding algorithm using a stack. *IBM J. Res. Dev.*, 13(6):675–685, November 1969.

[6] A.R. Murugan, H. El-Gamal, M.-O. Damen, and G. Caire. A unified framework for tree search decoding: Rediscovering the sequential decoder. *IEEE Transactions on Information Theory*, 52(3):933–953, March 2006.

[7] E. Viterbo and E. Biglieri. A universal decoding algorithm for lattice codes. In *Quatorzieme colloque GRETSI*, 1993.

[8] B. Hassibi and H. Vikalo. On the expected complexity of sphere decoding. In *Proceedings of Asilomar Conference on Signals, Systems and Computers*, volume 2, pages 1051–1055, November 2001.

[9] S.R. Kim, S. Lee, D.S. Kwon, S.K. Oh, and H.G. HAN. Sphere decoder and decoding method thereof, 2008. US Patent App. 11/722,131.

[10] M.-O. Damen, H. El-Gamal, and G. Caire. On maximum-likelihood detection and the search for the closest lattice point. *IEEE Transactions on Information Theory*, 49(10):2389–2402, October 2003.

[11] M. Pohst. On the computation of lattice vectors of minimal length, successive minima and reduced basis with applications. *ACM SIGSAM Bulletin*, 15:37–44, 1981.

[12] R. Kannan. Improved algorithms for integer programming and related lattice problems. *Proceedings of the ACM Symposium on Theory of computing*, pages 193–206, 1983.

[13] U. Fincke and M. Pohst. Improved methods for calculating vectors of short length in a lattice, including a complexity analysis. *Mathematics of Computation*, 44(170):463–471, 1985.

[14] P. M. Knuth and C. G. Lekkerkerker. Geometry of numbers. In *2nd edition Amsterdam*, 1987.

[15] G. Rekaya-Ben Othman. Nouvelles constructions algébriques de codes spatio-temporel atteignant le compromis multiplexage-diversité. In *Ph.D dissertation*, 2004.