Reduced-Complexity ML decodable STBCs: Revisited Design Criteria

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***Abstract*—In this work, we revisit the structure of weight matrices for Linear Dispersion STBCs to admit ML decoding with low-complexity. We first propose novel sufficient design criteria for linear STBCs considering an arbitrary number of antennas and an arbitrary coding rate. Then we apply the derived criteria to three families of codes, *multi-group decodable*, *fast decodable*, and *fast-group decodable* codes. We provide analytical proofs showing that the ML-decoding complexity of such codes depends only on the weight matrices and their ordering and not on the channel gains or the number of antennas and explaining why the so far used Hurwitz-Radon theory-based approaches do not exactly determine the complexity of all classes of STBCs under ML decoding.**

Keywords—Low-complexity decoding, maximum Likelihood decoding, fast decodable, Space-Time Block codes.

I. Introduction and Preliminaries

We consider in this work transmission over a coherent block-fading MIMO channel using n_t transmit and n_r receive antennas and coded modulations using length-T linear Space-Time Block Codes. The channel output is written as:

$$\mathbf{Y} = \mathbf{HX} + \mathbf{Z} \tag{1}$$

where $\mathbf{X} \in \mathbb{C}^{n_t \times T}$ is the codeword matrix sent over T channel uses and belonging to a codebook $\mathcal{C}$. $\mathbf{Z} \in \mathbb{C}^{n_r \times T}$ represents a complex-valued AWGN of i.i.d. entries of variance N_0 per real-valued dimension. The channel fadings are represented by the matrix $\mathbf{H} \in \mathbb{C}^{n_r \times n_t}$. We consider coherent transmission where the channel matrix $\mathbf{H}$ is assumed to be perfectly known (estimated) at the receiver. In addition, the fadings h_{ij} are assumed to be complex circularly symmetric Gaussian random variables of zero-mean and unit variance.

As linear Space-Time Block Codes (STBCs) are concerned within this work, the used STBC encodes κ complex information symbols represented by the complex-valued symbols vector $\mathbf{s} = [s_1, ..., s_\kappa]^t$ and the codeword matrix admits a Linear Dispersion (LD) decomposition form according to:

$$\mathbf{X} = \sum_{i=1}^{\kappa} \left(\Re(s_i)\mathbf{A}_{2i-1} + \Im(s_i)\mathbf{A}_{2i} \right) \tag{2}$$

where $\Re(s_i)$ and $\Im(s_i)$ correspond respectively to the real and imaginary parts of the κ complex information symbols and matrices $\mathbf{A}_l, l = 1, ..., 2\kappa$ are fixed $n_t \times T$ complex linearly independent matrices defining the code, known as LD or weight matrices. The rate of such codes is equal to $\frac{\kappa}{T}$ complex symbols per channel realization. When full rate codes are used, $\kappa = n_t T$. Moreover, we consider in this work 2^q-QAM constellations with q bits per symbol and for which the real and imaginary parts of the information symbols belong to a PAM modulation taking values in the set $[-(q-1), ..., (q-1)]$. In this work we are interested in the decoding of linear STBCs using Maximum Likelihood criterion. Accordingly, the receiver seeks an estimate $\hat{\mathbf{X}}$ of the transmitted codeword $\mathbf{X}$ by solving the minimization problem given by:

$$\hat{\mathbf{X}} = \underset{\mathbf{X} \in \mathcal{C}}{\operatorname{argmin}} \parallel \mathbf{Y} - \mathbf{HX} \parallel^2 \tag{3}$$

ML decoding remains thus to find the codeword matrix that minimizes the squared norm $m(\mathbf{X}) = \parallel \mathbf{Y} - \mathbf{HX} \parallel^2$. The complexity of ML decoding is determined by the minimum number of values of $m(\mathbf{X})$ that needs to be computed to find the ML solution. It is upper bounded by 2^{qTn_t}, the complexity of the exhaustive search. One way to avoid the high complexity of the exhaustive search consists in applying sequential decoders, such as the Sphere Decoder (SD) [1]. We are interested in this work in linear STBCs that admit low-complexity ML decoding using, without loss of generality, the sphere decoder.

Constructions of such codes date back to the *Complex Orthogonal* designs with the Alamouti code [2] and subsequent codes proposed in [3]. This family of codes offers the least ML decoding complexity that is linear as function of the constellation size. Their main drawback is their low maximum rate. *Quasi-Orthogonal* codes with full diversity and larger rates than the orthogonal designs have been later on proposed in [4]. Recently, 3 main families of ML-decodable codes with low-complexity have been discovered: *multi-group decodable* [5], *fast decodable* [6, 7], and *fast group decodable* codes [8]. A sub-class of fast decodable codes, termed *Block-orthogonal codes* has been proposed in [9]. Information symbols in such families of codes can be grouped into different partitions and decoded separately resulting in low-decoding complexity.

The construction and study of the above mentioned families of codes has been performed based on the so-called *Hurwitz-Radon Theory* (HR) to derive sufficient design criteria and conditions on the mutual orthogonality between the weight matrices defining the linear code. This theory has been later on used, in recent works, to define a second Quadratic Form approach in [10]. The sphere decoding complexity of linear STBCs is captured, under this approach, by a Hurwitz Radon Quadratic Form (HRQF) matrix. It is shown in [10] that the Quadratic Form approach allows to determine the SD complexity of the codes that belong to the families of multi-group decodable, fast decodable and fast-group decodable. Nevertheless, as highlighted in [10], it does not capture the class of block-orthogonal codes. In this work, we revisit the design of the weight matrices for STBCs to admit low-complexity ML decoding. Our contributions are as follows:

- We propose novel sufficient design criteria for reduced-complexity ML decodable linear STBCs considering an arbitrary number of antennas and an arbitrary coding rate.
- We apply the derived criteria to the families of multi-

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group decodable, fast decodable and fast group decodable codes showing that the SD complexity depends only on the weight matrices and their ordering and not on the channel gains.

- We provide analytical proofs explaining why the HRQF-based approach does not allow to capture exactly the SD complexity of all classes of STBCs and show that our criteria capture the case of Block-Orthogonal codes.

Examples, explanations and proofs, which have been ommited here due to space limitations, are provided in an expanded version of this paper in [11].

The remaining of this work is organized as follows: in section II we introduce the system model and review the formal definitions of the main classes of low-complexity ML decoding codes. In section III, we derive novel sufficient design criteria for SD of STBCs, apply them to the main 3 families of codes and show analytically the suboptimality of the sufficient design conditions existing in literature based on HR theory. Results of this work are summarized in a concluding section.

Notations: In this work we use the following notations: boldface letters are used for column vectors and capital boldface letters for matrices. Superscripts ^{t}, ^{H} and * denote transposition, Hermitian transposition, and complex conjugation, respectively. $\mathbb{Z}$ and $\mathbb{C}$ denote respectively the ring of rational integers and the field of complex numbers. i is the complex number such that $i^2 = -1$. In addition, $\mathbf{I}_n$ denotes the $n \times n$ identity matrix. Furthermore, for a complex number x, we define the $\tilde{(.)}$ operator from $\mathbb{C}$ to $\mathbb{R}^2$ as $\tilde{x} = [\Re(x), \Im(x)]^t$ where $\Re(.)$ and $\Im(.)$ denote real and imaginary parts. This operator can also be extended to a complex vector $\mathbf{x} = [x_1, ..., x_n]^t \in \mathbb{C}^n$ according to: $\tilde{\mathbf{x}} = [\Re(x_1), \Im(x_1), ..., \Re(x_n), \Im(x_n)]$. Also, we define the operator $\check{(.)}$ from $\mathbb{C}$ to $\mathbb{R}^{2\times 2}$ as: $\check{x} \triangleq \begin{bmatrix} \Re(x) & -\Im(x) \\ \Im(x) & \Re(x) \end{bmatrix}$. The operator $\check{(.)}$ can be in a similar way extended to $n \times n$ matrices by applying it to all the entries of the matrix which results in a $2n \times 2n$ real-valued matrix. We define also the $\text{vec}(.)$ operator that stacks the m columns of an $n \times m$ complex-valued matrix into an mn complex column vector. The $\| \, . \, \|$ operator denotes the Euclidean norm of a vector. We define also, for a complex number $x \in \mathbb{C}$ such that $x = \Re(x) + i\Im(x)$ the trace form such that $\text{Tr}(x) = \text{Tr}_{\mathbb{Q}(i)/\mathbb{Q}}(x) = 2\Re(x)$.

II. System Model, SD Complexity and Main Classes of Low-Complexity ML decoding codes

A. ML decoding problem

ML decoding using the Sphere Decoder exploits the triangular structure of the ML metric. In order to obtain this triangular structure, the complex-valued system in Eq.(1) is transformed into a real-valued one using the vectorization operator $\text{vec}(.)$ and the complex-to-real transformations $\tilde{(.)}$ and $\check{(.)}$. We obtain accordingly:

$$\tilde{\text{vec}}\,(\mathbf{Y}) = \mathbf{H}_{eq}\tilde{\mathbf{s}} + \tilde{\text{vec}}\,(\mathbf{Z}) \tag{4}$$

where $\mathbf{H}_{eq} \in \mathbb{R}^{2n_rT\times 2\kappa}$ is given by: $\mathbf{H}_{eq} = \left(\mathbf{I}_T \otimes \check{\mathbf{H}}\right)\mathbf{G}$ and its column vectors are given by $\mathbf{h}_i^{eq}, i =, ..., 2\kappa$. The real-valued matrix $\mathbf{G} \in \mathbb{R}^{2n_tT\times 2\kappa}$, termed a generator matrix of the linear code satisfies $\tilde{\text{vec}}(\mathbf{X}) = \mathbf{G}\tilde{\mathbf{s}}$ and can be written as function of the weight matrices as:

$$\mathbf{G} = [\tilde{\text{vec}}\,(\mathbf{A}_1)\,|\tilde{\text{vec}}\,(\mathbf{A}_2)\,|...|\tilde{\text{vec}}\,(\mathbf{A}_{2\kappa})] \tag{5}$$

Given that the ordering of the weight matrices in the LD form corresponds to the order of the information symbols as $\Re(s_1), \Im(s_1), ..., \Re(s_{2\kappa}), \Im(s_{2\kappa})$ which corresponds to the considered order in the complex-to-real transformation using the operator $\tilde{(.)}$, any change of the ordering of the information symbols results in a similar modification in the ordering of the weight matrices. The obtained real-valued system can then be written in the form:

$$\mathbf{y} = \mathbf{H}_{eq}\tilde{\mathbf{s}} + \mathbf{z} \tag{6}$$

Using this equivalent system, the ML decoding metric is equivalently written by:

$$m(\tilde{\mathbf{s}}) = \| \mathbf{y} - \mathbf{H}_{eq}\tilde{\mathbf{s}} \|^2 = \| \mathbf{Q}^t\mathbf{y} - \mathbf{R}\tilde{\mathbf{s}} \|^2 \tag{7}$$

where $\mathbf{Q} \in \mathbb{R}^{2n_rT\times 2\kappa}$ is orthogonal and $\mathbf{R} \in \mathbb{R}^{2\kappa\times 2\kappa}$ is an upper triangular matrix obtained both from the QR decomposition of the equivalent channel matrix $\mathbf{H}_{eq} = \mathbf{QR}$. Using Gram-Schmidt orthogonalization, matrices $\mathbf{Q}$ and $\mathbf{R}$ are given by: $\mathbf{Q} = [\mathbf{q}_1|\mathbf{q}_2|...|\mathbf{q}_{2\kappa}]$ where $\mathbf{q}_i, i = 1, ..., 2\kappa$ are column vectors and:

$$\mathbf{R} = \begin{bmatrix} \| \mathbf{r}_1 \| & < \mathbf{q}_1, \mathbf{h}_2^{eq} > & \cdots & < \mathbf{q}_1, \mathbf{h}_{2\kappa}^{eq} > \\ 0 & \| \mathbf{r}_2 \| & \cdots & < \mathbf{q}_2, \mathbf{h}_{2\kappa}^{eq} > \\ \vdots & \vdots & \ddots & \cdots \\ 0 & 0 & \cdots & \| \mathbf{r}_{2\kappa} \| \end{bmatrix}$$

where $\mathbf{r}_1 = \mathbf{h}_1^{eq}, \mathbf{q}_1 = \frac{\mathbf{r}_1}{\|\mathbf{r}_1\|}$ and for $i = 2, ..., 2\kappa, \mathbf{r}_i = \mathbf{h}_i^{eq} - \sum_{j=1}^{i-1} < \mathbf{q}_j^{eq}, \mathbf{h}_i^{eq} > \mathbf{q}_j$, $\mathbf{q}_i = \frac{\mathbf{r}_i}{\|\mathbf{r}_i\|}$.

When the Sphere Decoder is used to solve this minimization problem, its complexity can be alleviated thanks to zero entries in the matrix $\mathbf{R}$, that depend on the used code and the ordering of the real and imaginary parts of the symbols in the vector $\tilde{\mathbf{s}}$, and accordingly the ordering of the weight matrices in the generator matrix $\mathbf{G}$. In literature, we distinguish 3 main classifications of codes using which the matrix $\mathbf{R}$ has an interesting form enabling reduced-complexity ML decoding: *Fast decodable codes*, *Multi-group decodable codes* and *Fast group decodable codes*. By structure, it is meant the locations of the zero entries R_{ij}. The construction of such codes and determination of the structure of the matrix $\mathbf{R}$ for these classes is, commonly in literature, determined using a *mutual orthogonality* property of the weight matrices based on which two main approaches are proposed in literature: *Hurwitz-Radon* theory (HR) approach and a *Quadratic Form* (HRQF) approach. We detail in the following subsections these two approaches and summarize the main results. We provide for convenience the definition of a partition as follows.

Definition 1 (A set partition)**.** *We call a partition $\{a_1, ..., a_n\}$ into g non-empty subsets $\Gamma_1, ..., \Gamma_g$ with cardinalities $K_1, ..., K_g$ an ordered partition if $\{a_1, ..., a_{K_1}\} \in \Gamma_1$, $\{a_{K_1+1}, ..., a_{K_1+K_2}\} \in \Gamma_2$, so on until $\left\{a_{\sum_{i=1}^{g-1} K_i+1}, ..., a_{\sum_{i=1}^{g} K_i}\right\} \in \Gamma_g$.*

B. Hurwitz-Radon theory-based approach

The HR theory-based approach uses in its essence the mutual orthogonality of weight matrices [12]. The main result is stated in the following theorem [7].

Theorem 1. *For an STBC with κ independent complex information symbols and 2κ linearly independent matrices $\mathbf{A}_l, l = 1, ..., 2\kappa$, if, for any i and j, $i \neq j$, $1 \leq i \neq j \leq 2\kappa$, $\mathbf{A}_i\mathbf{A}_j^H + \mathbf{A}_j\mathbf{A}_i^H = \mathbf{0}_{n_t}$, then the i^{th} and j^{th} columns of the equivalent channel matrix $\mathbf{H}_{eq}$ are orthogonal.*

This property has been used to define and construct particular classes of codes defined below.

Definition 2 (*Multi-group decodable codes*)**.** *An STBC is said to be $g-$group decodable if there exists a partition of $\{1,2,...,2\kappa\}$ into g non-empty subsets $\Gamma_1,\Gamma_2,...,\Gamma_g$ such that $\mathbf{A}_l\mathbf{A}_m^H+\mathbf{A}_m\mathbf{A}_l^H=\mathbf{0}$, whenever $l\in\Gamma_i$ and $m\in\Gamma_j$ and $i\neq j$. The corresponding $\mathbf{R}$ matrix has the following form:*

$$\mathbf{R}=\begin{bmatrix}\Delta_1 & \mathbf{0} & \cdots & \mathbf{0}\\ \mathbf{0} & \Delta_2 & \cdots & \mathbf{0}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{0} & \mathbf{0} & \cdots & \Delta_g\end{bmatrix} \tag{8}$$

where $\Delta_i, i=1,...,g$ is a square upper triangular matrix.

Definition 3 (*Fast decodable codes*)**.** *An STBC is said to be fast Sphere Decodable code if there exists a partition of $\{1,2,...,L\}$ where $L\leq 2\kappa$ into g non-empty subsets $\Gamma_1,\Gamma_2,...,\Gamma_g$ such that $<\mathbf{q}_i,\mathbf{h}_j^{eq}>=0,(i<j)$, whenever $i\in\Gamma_p$ and $j\in\Gamma_q$ and $p\neq q$. The $\mathbf{R}$ matrix corresponding to fast decodable codes has the form:*

$$\mathbf{R}=\begin{bmatrix}\Delta & \mathbf{B}_1\\ \mathbf{0} & \mathbf{B}_2\end{bmatrix} \tag{9}$$

where Δ is an $L\times L$ block diagonal, upper triangular matrix, $\mathbf{B}_1$ is a rectangular matrix and $\mathbf{B}_2$ is a square upper triangular matrix.

Definition 4 (*Fast-group decodable codes*)**.** *An STBC with weight matrices $\mathbf{A}_l, l=1,...,2\kappa$ is said to be fast group decodable if it satisfies the following conditions:*

- *There exists a partition of $\{1,...,2\kappa\}$ into g non-empty subsets $\Gamma_1,...,\Gamma_g$ such that $\mathbf{A}_l\mathbf{A}_m^H+\mathbf{A}_m\mathbf{A}_l^H=\mathbf{0}_{n_t}$ for $l\in\Gamma_i$, $m\in\Gamma_j$ and $i\neq j$.*
- *In any partition Γ_i, $<\mathbf{q}_{i_{l_1}},\mathbf{h}_{i_{l_2}}^{eq}>=0$ $(l_1=1,2,...,L_i-1$ and $l_2=l_1+1,...,L_i)$ and $L_i\leq|\Gamma_i|$ where $i=1,2,...,g$.*

The corresponding $\mathbf{R}$ matrix has the form:

$$\mathbf{R}=\begin{bmatrix}\mathbf{R}_1 & \mathbf{0} & \cdots & \mathbf{0}\\ \mathbf{0} & \mathbf{R}_2 & \cdots & \mathbf{0}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{R}_g\end{bmatrix} \tag{10}$$

where at least one $\mathbf{R}_i, i=1,...,g$ has the fast-decodability form.

In addition to these families of codes, recently, a particular sub-class of fast decodable codes has been proposed, termed *Block-Orthogonal codes*. Several known codes in literature belong to this family of codes, such as the BHV [6] code, Srinath-Rajan code [7] and codes from Cyclic Division algebras [13]. The formal definition of the sufficient design criteria for Block-Orthogonal codes were first given in [9] for codes with parameters $(\Gamma,k,1)$ and recently generalized in [14, 15] for codes with parameters (Γ,k,γ) for arbitrary sizes of sub-blocks considering the matrices $\mathbf{R}_i, i=1,...,\Gamma$ having the same size of sub-blocks. These sufficient design conditions are summarized in the following lemma [14, 15].

Lemma 2. *Let the $\mathbf{R}$ matrix of an STBC with weight matrices $\{\mathbf{A}_1,...,\mathbf{A}_L\}$ and $\{\mathbf{B}_1,...,\mathbf{B}_l\}$ be $\mathbf{R}=\begin{bmatrix}\mathbf{R}_1 & \mathbf{E}\\ \mathbf{0} & \mathbf{R}_2\end{bmatrix}$, where $\mathbf{R}_1$ is an $L\times L$ upper triangular block-orthogonal matrix with parameters $(\Gamma-1,k,\gamma)$, $\mathbf{E}$ is an $L\times l$ matrix and $\mathbf{R}_2$ is an $l\times l$ upper triangular matrix. The STBC will be block orthogonal with parameters (Γ,k,γ) if the following conditions are satisfied:*

- *The matrices $\{\mathbf{B}_1,...,\mathbf{B}_l\}$ are $k-$group decodable with γ variables in each group.*
- *The matrices $\{\mathbf{A}_1,...,\mathbf{A}_L\}$ when used as weight matrices for an STBC yield an $\mathbf{R}$ matrix having a block orthogonal structure with parameters $(\Gamma-1,k,\gamma)$. When $\Gamma=2$, then $L=l$ and the matrices $\{\mathbf{A}_1,...,\mathbf{A}_L\}$ are $k-$group decodable with variables γ in each group.*
- *The set of matrices $\{\mathbf{A}_1,...,\mathbf{A}_L,\mathbf{B}_1,...,\mathbf{B}_l\}$ are such that the matrix $\mathbf{R}$ obtained is of full rank.*
- *The matrix $\mathbf{E}^t\mathbf{E}$ is a block diagonal matrix with k blocks of size $\gamma\times\gamma$.*

$\mathbf{R}$ matrix for Block Orthogonal codes has the form:

$$\mathbf{R}=\begin{bmatrix}\mathbf{R}_1 & \mathbf{B}_{12} & \cdots & \mathbf{B}_{1\Gamma}\\ \mathbf{0} & \mathbf{R}_2 & \cdots & \mathbf{B}_{2\Gamma}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{R}_\Gamma\end{bmatrix} \tag{11}$$

where each matrix $\mathbf{R}_i, i=1,...,\Gamma$ is full rank, block diagonal, upper triangular with k blocks $\mathbf{U}_{i1},...,\mathbf{U}_{ik}$ each of size $\gamma\times\gamma$ and $\mathbf{B}_{ij}, i=1,...,\Gamma, j=i+1,...,\Gamma$ are non-zero matrices.

C. Quadratic Form-based approach

In theorem 1, it is shown that the Hurwitz-Radon Theory capturing the orthogonality between two weight matrices is sufficient to obtain orthogonality of corresponding columns of the equivalent channel matrix. This property was later on used in [10] to develop a Quadratic Form termed *Hurwitz Radon Quadratic Form* (HRQF). This quadratic form has been considered before in literature [16] to determine whether Quaternion algebras or Biquaternion algebras are division algebras. In [10], this quadratic form is further exploited to define the zero structure of the matrix $\mathbf{R}$ by associating to the HRQF a matrix $\mathbf{U}$ such that $U_{ij}=\|\mathbf{A}_i\mathbf{A}_j^H+\mathbf{A}_j\mathbf{A}_i^H\|^2$ and $U_{ij}=0$ if and only if $\mathbf{A}_i\mathbf{A}_j^H+\mathbf{A}_j\mathbf{A}_i^H=\mathbf{0}_{n_t}$. This form has been used in [10] to determine sufficient conditions for an STBC to admit multi-group, fast and fast-group decodability as summarized in the following lemmas [10].

Lemma 3. *Let an STBC with κ independent complex symbols, 2κ weight matrices and HRQF matrix $\mathbf{U}$. If there exists an ordered partition of $\{1,2,...,2\kappa\}$ into g non-empty subsets $\Gamma_1,...,\Gamma_g$ such that $U_{ij}=0$ whenever $i\in\Gamma_p$ and $j\in\Gamma_q$ and $p\neq q$, then the code is $g-$group sphere decodable.*

Lemma 4. *Let an STBC with κ independent complex symbols, 2κ weight matrices and HRQF matrix $\mathbf{U}$. If there exists a partition of $\{1,2,...,L\}$ where $L\leq 2\kappa$ into g non-empty subsets $\Gamma_1,...,\Gamma_g$ such that $U_{ij}=0$ whenever $i\in\Gamma_p$ and $j\in\Gamma_q$ and $p\neq q$, then the code is fast decodable.*

Lemma 5. *Let an STBC with κ independent complex symbols, 2κ weight matrices and HRQF matrix $\mathbf{U}$. If there exists a partition of $\{1,2,...,L\}$ where $L\leq 2\kappa$ into g non-empty subsets $\Gamma_1,...,\Gamma_g$ with cardinalities $\kappa_1,...,\kappa_g$ such that $U_{ij}=0$ whenever $i\in\Gamma_p$ and $j\in\Gamma_q$ and $p\neq q$, and if any group Γ_i admits fast decodability, then the code is fast group decodable.*

These lemmas state that the HRQF matrix totally determines the fast sphere decodability of STBCs and provide sufficient conditions for an STBC to be multi-group, fast or fast-group decodable. Nevertheless, as highlighted in [10], in some cases, for instance in the case of Block orthogonal codes, the HRQF approach does not capture the zero structure of the $\mathbf{R}$ matrix. In such cases, it is possible to have entries $R_{ij} \neq 0$ even if the corresponding weight matrices $\mathbf{A}_i$ and $\mathbf{A}_j$ are HR orthogonal which is equivalent to have the corresponding entry in the HRQF matrix $U_{ij} = 0$. Authors in [10] do not provide an explanation for having such configurations.

III. Sufficient Design Criteria for Low-Complexity ML Decoding of STBCs

In this work, we first aim to provide sufficient conditions on the structure of the weight matrices for an STBC that fully determine the SD ML-decoding complexity of any STBC. These design criteria are stated in theorem 6.

Theorem 6. *For an STBC with k independent complex symbols and $2k$ weight matrices $\mathbf{A}_l$ for $l = 1, ..., 2\kappa$, if for any i and j, $i \neq j, 1 \leq i, j \leq 2\kappa$ for all $l = 1, ..., T$* ***one or both*** *of the following conditions is satisfied:*

$$\mathrm{Tr}\left(a_{ql}^{(i)}\left(a_{pl}^{(j)}\right)^* + a_{pl}^{(i)}\left(a_{ql}^{(j)}\right)^*\right) = 0\,,\ \forall q = 1, ..., n_t, p = q, q{+}1, ..., n_t$$

$$\mathrm{Tr}\left(i\left[a_{ql}^{(i)}\left(a_{pl}^{(j)}\right)^* - a_{pl}^{(i)}\left(a_{ql}^{(j)}\right)^*\right]\right) = 0\,,\ \forall q = 1, ..., n_t, p = q{+}1, ..., n_t$$

where $a_{ql}^{(i)}$ (resp. $a_{pl}^{(j)}$) is the entry of the matrix $\mathbf{A}_i$ (resp. $\mathbf{A}_j$) at row q and column l (resp. at row p and column l), then the i^{th} and j^{th} columns of the equivalent channel matrix $\mathbf{H}_{eq}$ are orthogonal. ***Both conditions hold at the same time if and only if*** $a_{ql}^{(i)} = 0$ ***or*** $a_{pl}^{(j)} = 0$.

Proof: We know from the expression of the equivalent channel matrix that: $< \mathbf{h}_i^{eq}, \mathbf{h}_j^{eq} >= \sum_{l=1}^{T} \tilde{\mathbf{a}}_{il}^t \check{\mathbf{H}}^t \check{\mathbf{H}} \tilde{\mathbf{a}}_{jl} = \sum_{l=1}^{T} \tilde{\mathbf{a}}_{il}^t \mathbf{M} \tilde{\mathbf{a}}_{jl}$ where $\mathbf{a}_{il}, \mathbf{a}_{jl}, l = 1, ..., T$ are the l^{th} columns of respectively the weight matrix $\mathbf{A}_i$ and $\mathbf{A}_j$. Using the operator $(\check{.})$, we show that the matrix $\mathbf{M}$ is symmetric and satistifies the following properties:

$$M_{2i-1,2i-1} = M_{2i,2i}, M_{2i-1,2i} = 0\ ,\ \forall i = 1, ..., n_t$$
$$M_{2i-1,2j-1} = M_{2i,2j}, M_{2i,2j-1} = -M_{2i-1,2j}\ ,\ \forall 1 \leq i < j \leq n_t$$

Let $T_l = \tilde{\mathbf{a}}_{il}^t \mathbf{M} \tilde{\mathbf{a}}_{jl}$. By deriving the computation and using the properties of the matrix $\mathbf{M}$, we show that $T_l = A_l + B_l$ where:

$$A_l = \frac{1}{2}\sum_{q=1}^{n_t} M_{2q-1,2q-1}\mathrm{Tr}\left(a_{ql}^{(i)}\left(a_{ql}^{(j)}\right)^*\right) + \frac{1}{2}\sum_{q=1}^{n_t}\sum_{p=q+1}^{n_t} M_{2q-1,2p-1}\mathrm{Tr}\left(a_{ql}^{(i)}\left(a_{pl}^{(j)}\right)^* + a_{pl}^{(i)}\left(a_{ql}^{(j)}\right)^*\right)$$

$$B_l = \frac{1}{2}\sum_{q=1}^{n_t}\sum_{p=q+1}^{n_t} M_{2q-1,2p}\mathrm{Tr}\left(i\left[a_{ql}^{(i)}\left(a_{pl}^{(j)}\right)^* - a_{pl}^{(i)}\left(a_{ql}^{(j)}\right)^*\right]\right)$$

Notice that the terms $\mathrm{Tr}\left(a_{ql}^{(i)}\left(a_{pl}^{(j)}\right)^*\right)$ in A_l and $\mathrm{Tr}\left(ia_{ql}^{(i)}\left(a_{pl}^{(j)}\right)^*\right)$ in B_l are equal to zeros at the same time if and only if either $a_{ql}^{(i)} = 0$ or $a_{pl}^{(j)}$. The proof follows given $< \mathbf{h}_i^{eq}, \mathbf{h}_j^{eq} >= \sum_{l=1}^{T} T_l$. Details of the proof are ommited for space limitations and provided in the long version of this paper [11]. ■

Theorem 6 states a component-wise mutual orthogonality criterion involving the entries of weight matrices corresponding to column vectors of the equivalent channel matrix. We provide in the following lemma an analytical proof that shows the suboptimality of the HR theory and HRQF-based approaches and explains why the conditions proposed in these approaches do not capture all the families of low-complexity ML decoding STBCs in contrast to our derived sufficient design conditions in Theorem 6.

Lemma 7. *Consider an STBC with κ independent complex information symbols and 2κ weight matrices $\mathbf{A}_l, l = 1, ..., 2\kappa$. If for any i and j, $1 \leq i, j \leq 2k$ the matrices $\mathbf{A}_i$ and $\mathbf{A}_j$ are mutually orthogonal, i.e. satisfy $\mathbf{A}_i\mathbf{A}_j^H + \mathbf{A}_j\mathbf{A}_i^H = \mathbf{0}_{n_t}$, then $\forall p = 1, ..., n_t$, $\sum_{l=1}^{n_t}\mathrm{Tr}\left(a_{pl}^{(i)}\left(a_{pl}^{(j)}\right)^*\right) = 0$.*

Proof:

We know from the properties of the $(\check{.})$ operator that $\mathbf{A} = \mathbf{BC} \Leftrightarrow \check{\mathbf{A}} = \check{\mathbf{B}}\check{\mathbf{C}}$. Using this property, we have $\mathbf{A}_i\mathbf{A}_j^H + \mathbf{A}_j\mathbf{A}_i^H = \mathbf{0}_{n_t} \Leftrightarrow \check{\mathbf{A}}_i\left(\check{\mathbf{A}}_j\right)^t + \check{\mathbf{A}}_j\left(\check{\mathbf{A}}_i\right)^t = \mathbf{O}_{2n_t}$. This means that the matrix $\mathbf{V} = \check{\mathbf{A}}_i\left(\check{\mathbf{A}}_j\right)^t$ is skew-symmetric, thus its diagonal elements are zeros. Using the complex-to-real transformation, it is easy to show that the diagonal entries are given, $\forall p = 1, ..., n_t$ by:

$$V_{2p-1,2p-1} = V_{2p,2p} = \sum_{l=1}^{n_t}\left(\Re\left(a_{pl}^{(i)}\right)\Re\left(a_{pl}^{(j)}\right) + \Im\left(a_{pl}^{(i)}\right)\Im\left(a_{pl}^{(j)}\right)\right) = \sum_{l=1}^{n_t}\left(\mathrm{Tr}\left(a_{pl}^{(i)}\left(a_{pl}^{(j)}\right)^*\right)\right)$$

Having diagonal entries of $\mathbf{V}$ equal to 0 ends the proof. ■

From Lemma 7, we can easily see that the sufficient condition for having orthogonality between two columns in the equivalent channel matrix as proposed using the HR mutual orthogonality and HRQF approach captures only the summation of the trace forms of the components $a_{pl}^{(i)}\left(a_{pl}^{(j)}\right)^*$ and imposes that this summation be zero for all $p = 1, ..., n_t$. However, as proved in Theorem 6, in order to have the i^{th} and j^{th} columns of $\mathbf{H}_{eq}$ orthogonal, it is sufficient to have the individual trace forms for $a_{pl}^{(i)}\left(a_{pl}^{(j)}\right)^*$ or $ia_{pl}^{(i)}\left(a_{pl}^{(j)}\right)^*$ equal to 0 and not the summation of the trace forms be 0. Of course when the individual trace forms are null, the summation is also equal to 0, however, if the summation is equal to 0, the individual trace forms can be different of 0. In such cases, the HR mutual orthogonality is satisfied and the entry U_{ij} of the HRQF matrix is equal to 0 without having orthogonality of columns i and j of the equivalent channel matrix and thus the corresponding entry in the $\mathbf{R}$ matrix $R_{ij} \neq 0$.

Having derived the sufficient design criteria for having orthogonality of columns in the equivalent channel matrix, we move now to the application of these conditions and show that these criteria are enough to determine the SD complexity of an STBC, prove that the SD complexity depends only on the weight matrices and their ordering and not on the channel matrix or the number of receive antennas. We start with the class of multi-group decodable codes.

Lemma 8. *Let an STBC with κ independent complex symbols and 2κ weight matrices. If there exists an ordered partition of $\{1, 2, ..., 2\kappa\}$ into g non-empty subsets $\Gamma_1, ..., \Gamma_g$ such that for all $l = 1, ..., T$ at least one of the following conditions is satisfied:*

$$\mathrm{Tr}\left(a_{ql}^{(i)}\left(a_{pl}^{(j)}\right)^* + a_{pl}^{(i)}\left(a_{ql}^{(j)}\right)^*\right) = 0\,,\ \forall q = 1, ..., n_t, p = q, q{+}1, ..., n_t$$

$$\mathrm{Tr}\left(i\left[a_{ql}^{(i)}\left(a_{pl}^{(j)}\right)^* - a_{pl}^{(i)}\left(a_{ql}^{(j)}\right)^*\right]\right) = 0\,,\ \forall q = 1, ..., n_t, p = q{+}1, ..., n_t$$

whenever $i \in \Gamma_m$ and $j \in \Gamma_n$ and $m \neq n$, then the code is $g-$group sphere decodable.

Lemma 9. *Let an STBC with κ independent complex symbols and 2κ weight matrices. If there exists a partition of $\{1, 2, ..., L\}$ where $L \leq 2\kappa$ into g non-empty subsets $\Gamma_1, \Gamma_2, ..., \Gamma_g$ such that for all $l = 1, ..., T$ at least one of the following conditions is satisfied:*

$$\mathrm{Tr}\left(a_{ql}^{(i)}\left(a_{pl}^{(j)}\right)^* + a_{pl}^{(i)}\left(a_{ql}^{(j)}\right)^*\right) = 0\ ,\ \forall q = 1, ..., n_t, p = q, q{+}1, ..., n_t$$

$$\mathrm{Tr}\left(i\left[a_{ql}^{(i)}\left(a_{pl}^{(j)}\right)^* - a_{pl}^{(i)}\left(a_{ql}^{(j)}\right)^*\right]\right) = 0\ ,\ \forall q = 1, ..., n_t, p = q{+}1, ..., n_t$$

whenever $i \in \Gamma_m$ and $j \in \Gamma_n$ and $m \neq n$, then the code is fast decodable.

Lemma 10. *Let an STBC with κ independent complex symbols and 2κ weight matrices. If there exists a partition of $\{1, 2, ..., L\}$ where $L \leq 2\kappa$ into g non-empty subsets $\Gamma_1, \Gamma_2, ..., \Gamma_g$ with cardinalities $\kappa_1, ..., \kappa_g$ such that for all $l = 1, ..., T$ at least one of the following conditions holds:*

$$\mathrm{Tr}\left(a_{ql}^{(i)}\left(a_{pl}^{(j)}\right)^* + a_{pl}^{(i)}\left(a_{ql}^{(j)}\right)^*\right) = 0\ ,\ \forall q = 1, ..., n_t, p = q, q{+}1, ..., n_t$$

$$\mathrm{Tr}\left(i\left[a_{ql}^{(i)}\left(a_{pl}^{(j)}\right)^* - a_{pl}^{(i)}\left(a_{ql}^{(j)}\right)^*\right]\right) = 0\ ,\ \forall q = 1, ..., n_t, p = q{+}1, ..., n_t$$

whenever $i \in \Gamma_m$ and $j \in \Gamma_n$ and $m \neq n$, and if any group Γ_i admits fast decodability, then the code is fast group decodable.

The last class of codes studied in this work is the Block orthogonal family. We provide in the following sufficient design criteria for STBCs to be (Γ, k, γ)-Block Orthogonal.

Lemma 11. *Let the $\mathbf{R}$ matrix of an STBC with weight matrices $\{\mathbf{A}_1, ..., \mathbf{A}_L\}$ and $\{\mathbf{B}_1, ..., \mathbf{B}_l\}$ be $\mathbf{R} = \begin{bmatrix} \mathbf{R}_1 & \mathbf{E} \\ \mathbf{0} & \mathbf{R}_2 \end{bmatrix}$, where $\mathbf{R}_1$ is an $L \times L$ upper triangular block-orthogonal matrix with parameters $(\Gamma - 1, k, \gamma)$, $\mathbf{E}$ is an $L \times l$ matrix and $\mathbf{R}_2$ is an $l \times l$ upper triangular matrix. The STBC will be block orthogonal with parameters (Γ, k, γ) if the following conditions are satisfied:*

- *If there exists an ordered partition of the set of matrices $\{\mathbf{B}_1, ..., \mathbf{B}_l\}$ into k non-empty subsets $S_1, ..., S_k$ each of cardinality γ such that for all $l = 1, ..., T$ at least one of the following conditions is satisfied:*

  $$\mathrm{Tr}\left(a_{ql}^{(i)}\left(a_{pl}^{(j)}\right)^* + a_{pl}^{(i)}\left(a_{ql}^{(j)}\right)^*\right) = 0\ ,\ \forall q = 1, ..., n_t, p = q, q{+}1, ..., n_t$$

  $$\mathrm{Tr}\left(i\left[a_{ql}^{(i)}\left(a_{pl}^{(j)}\right)^* - a_{pl}^{(i)}\left(a_{ql}^{(j)}\right)^*\right]\right) = 0\ ,\ \forall q = 1, ..., n_t, p = q{+}1, ..., n_t$$

  whenever $i \in S_m$ and $j \in S_n$ and $m \neq n$.
- *The matrices $\{\mathbf{A}_1, ..., \mathbf{A}_L\}$ when used as weight matrices for an STBC yield an $\mathbf{R}$ having a block orthogonal structure with parameters $(\Gamma - 1, k, \gamma)$. When $\Gamma = 2$, then $L = l$ and the matrices $\{\mathbf{A}_1, ..., \mathbf{A}_L\}$ are $k-$group decodable with variables γ in each group.*
- *The set of matrices $\{\mathbf{A}_1, ..., \mathbf{A}_L, \mathbf{B}_1, ..., \mathbf{B}_l\}$ are such that the matrix $\mathbf{R}$ obtained is of full rank.*
- *The matrix $\mathbf{E}^t\mathbf{E}$ is a block diagonal matrix with k blocks of size $\gamma \times \gamma$.*

Detailed proofs and examples of the provided lemmas can be found in the long version of this paper.

IV. Conclusion

This work is dedicated to the design of STBCs that admit low-complexity ML decoding using sequential decoders. We proposed novel sufficient design criteria for weight matrices defining the Linear Dispersion code for an arbitrary number of antennas and any coding rate. Our criteria explain why the existing approaches fail in determining the zero structure for some families of codes, for instance Block Orthogonal codes and show that decoding complexity of STBCs depends only on the weight matrices and their ordering and not on the channel matrix.

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