MAP Decoder for Physical-Layer Network Coding Using Lattice Sphere Decoding

Asma Mejri, *Student Member, IEEE*, and Ghaya Rekaya-Ben Othman, *Member, IEEE*
Telecom ParisTech, 46 Rue Barrault, 75013 Paris, France, {amejri,rekaya}@telecom-paristech.fr

***Abstract*—We focus in this work on optimal *maximum a posteriori* (MAP) decoding for lattice-based physical-layer network coding operating in Gaussian multiple access relay channels. We consider a general lattice design that makes our results hold for any lattice coding schemes of any dimensions including the Compute-and-Forward framework. By examining the MAP decoding rule, we first derive an analytical bound on the codeword error probability taking into consideration decoding errors at the relay. Besides, we derive a novel MAP decoding metric using which we develop a novel, practical and easy-to-implement MAP decoding algorithm based on lattice sphere decoding. We further provide numerical results that demonstrate the effectiveness of our algorithm and show its outperformance over existing suboptimal decoders.**

***Index Terms*—Physical-layer network coding, lattice codes, maximum a posteriori decoding, sphere decoder.**

I. Introduction

Physical-layer network coding (PNC) is a new coding perspective that has fundamentally changed the management of interference in multi-user communication networks. Traditionally considered as a nuisance to be avoided, interference resulting from the superposition of data at intermediate relay nodes is exploited to decode combinations of original signals [1] and result in higher end-to-end transmission rates.

We are interested in this work in a very promising category of linear PNC joint to lattice-based channel coding. The first framework that falls into this class is the Compute-and-Forward (CF) scheme proposed by Nazer and Gastpar in [2]. Based on nested lattice coding structures, this new strategy allows to take advantage of the interference provided by the wireless medium to decode a *noiseless integer linear* combination of original codewords with higher rates. Channel decoding at intermediate relays is based on a standard minimum distance decoding. Under this assumption, a union bound estimate of the error probability at the relays was derived in [3], and we have addressed in [4] and [5] the end-to-end error performance evaluation in the multi-source relay channel and the two-way relay channel respectively. Later on, independently in [6] and [7], the Maximum Likelihood (ML) decoder was investigated. An algebraic extension of the CF using lattice partitions related to finitely generated modules over principal ideal domains was proposed by Feng *et al.* in [3]. For this scheme, minimum distance decoding is also considered. Wilson *et al.* investigated in [8] a lattice-based computation in the Gaussian multiple access channel based on the minimum angle decoder.

A main common limitation of the above mentioned schemes is the use of suboptimal decoders at the relay's level. The worthiness of using the maximum a posteriori (MAP) decoder, which is the optimal decoding criterion for this scheme, its gain over the existing suboptimal decoders as well as its implementation in practice are still not solved. Authors in [8] expect that the MAP decoder offers better performance than the existing decoders. However, no deeper investigations of the MAP's performance have been provided so far to check whether it is worth to apply this decoder or not. The novelty of our work is the analysis of the MAP decoder for lattice-based network coding strategies operating in the Gaussian multiple access relay channel. We consider a general lattice design that makes our results applicable to any lattice encoding schemes including the above cited works. After describing the considered lattice design and analyzing the MAP decoding rule in section II, our contributions are organized as follows:

- In section III, we derive a union bound estimate on the codeword error probability taking into account decoding errors at the relay from which we propose a lattice design criterion.
- In section IV, we derive a novel MAP decoding metric from which we develop a new and easy-to-implement decoding algorithm based on spherical lattice decoding.
- In section V, we provide numerical results that show that our proposed MAP algorithm outperforms the existing minimum distance decoding while maintaining the same complexity order.

Concluding remarks and future investigations are summarized in section VI.

II. System model and problem statement

A. Lattice Design

An $n-$dimensional real valued lattice Λ is a discrete additive subgroup of $\mathbb{R}^n$ given by $\Lambda = \{\mathbf{x} = \mathbf{M}\mathbf{s}, \mathbf{s} \in \mathbb{Z}^n\}$, where the matrix $\mathbf{M}$ refers to the generator matrix of Λ and is assumed to be full rank. A main property of a lattice structure is linearity, meaning that for any $\mathbf{x}, \mathbf{y} \in \Lambda, a, b \in \mathbb{Z}, a\mathbf{x}+b\mathbf{y} \in \Lambda$.

We consider in our analysis a general *lattice design* given by the pair $(\Lambda, \mathcal{R})$ where $\Lambda \subset \mathbb{R}^n$ is a lattice and $\mathcal{R}$ is a compact convex set of $\mathbb{R}^n$ containing the zero vector. $\mathcal{R}$ defines the shaping region of the lattice design and acts to satisfy the transmission power constraint. For this lattice design, we define the codebook $\mathcal{C} = \{\Lambda \cap \mathcal{R}\}$, the set of points obtained from the intersection of Λ with the shaping region $\mathcal{R}$. The considered design admits most of the lattice coding schemes proposed in literature including the nested lattice scheme of Nazer and Gastpar [2], the algebraic framework of Feng *et al.* in [3] and the lattice scheme considered by Wilson *et al.* in [8]. These structures mainly differ in the choice of the

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shaping region. For example, in the case of the CF scheme, $\mathcal{R}$ is designed using a shaping lattice Λ_{C} termed the coarse lattice such that $\Lambda_{\mathrm{C}} \subset \Lambda$. The generalization of our design makes our results straightforwardly applicable to all of the above mentioned schemes.

B. Computation in Gaussian Multiple Access Relay Channel

Consider a real $n-$dimensional lattice design $(\Lambda, \mathcal{R})$ and let $\mathbf{M}$ denote a generator matrix of the lattice Λ. The system model we are interested in within this work is the real-valued Gaussian multiple access relay channel composed of K source nodes that transmit simultaneously their data to a common relay R. In a first step, sources encode their information messages onto lattice codewords $\mathbf{x}_i$ drawn uniformally and independently from the same codebook $\mathcal{C}$ using a same lattice design $(\Lambda, \mathcal{R})$. Original codewords meet then the power constraint defined by the shaping region and satisfy: $\frac{1}{n}\mathbb{E}\left\{\parallel \mathbf{x}_i \parallel^2\right\} \leq P, \;\; P > 0, \; i = 1, ..., K$. The channel output is given by:

$$\mathbf{y} = \sum_{i=1}^{K} \mathbf{x}_i + \mathbf{z} \tag{1}$$

where $\mathbf{z} \in \mathbb{R}^n$ stands for the AWGN generated i.i.d according to a normal distribution $\mathcal{N}\left(0, \sigma^2 \mathbf{I}_n\right)$. We define the signal to noise ratio (SNR) as $\rho = \frac{P}{\sigma^2}$.

From this observed vector, the relay attempts to decode a noiseless sum of the original codewords: $\lambda = \sum_{i=1}^{K} \mathbf{x}_i$ which, given the linear structure of the lattice, will also be a codeword from Λ. This decoding objective can be considered as the first step of the Compute-and-Forward relaying strategy (since for this scheme, the ultimate goal of the relay is to recover the noiseless sum modulo the coarse lattice), or a Denoise-and-Forward technique [9].

The relay is equipped with a decoder $\mathcal{D}$ that generates an estimate $\hat{\lambda}$ of the desired sum. The codeword error probability at the relay corresponds to the decoding errors on the sum codewords. It is then given by $\mathrm{P_{error}} = \Pr\left(\hat{\lambda} \neq \lambda\right)$.

(a) Histogram for K=2. (b) Histogram for K=5.

Fig. 1. Histogram of the codebook induced by the sum of codewords.

Let Λ_{s} denote the *sum codebook* which is the set of all sum codewords λ. Given the linear structure of the lattice Λ, Λ_{s} is a subset of Λ restricted to a *sum shaping region* $\mathcal{R}_{\mathrm{s}}$ such that all sum codewords λ_{s} fall within this region. Given that the set Λ_{s} is obtained through a superposition of the originally transmitted codewords, its distribution is no longer uniform. This observation was first highlighted in [8]. As a proof of concept, we illustrate in Fig.1 two examples of the statistical distribution of a sum codebook resulting from the superposition of 2-dimensional lattice codewords for the case of $K = 2$ and $K = 5$ considering a lattice Λ of a generator matrix $\mathbf{M} = \begin{bmatrix} 2 & 3 \\ 3 & -1 \end{bmatrix}$.

Under this non-uniform distribution property, the optimal decoder at the relay that minimizes the probability of codeword error $\mathrm{P_{error}}$, is the *maximum a posteriori* decoder given according to the following

$$\begin{aligned}\hat{\lambda}_{\mathrm{map}} &= \underset{\lambda\in\Lambda_{\mathrm{s}}}{\operatorname{argmax}}\, p\left(\lambda|\mathbf{y}\right) = \underset{\lambda\in\Lambda_{\mathrm{s}}}{\operatorname{argmax}}\, p(\lambda) p\left(\mathbf{y}|\lambda\right) \\ &= \underset{\lambda\in\Lambda_{\mathrm{s}}}{\operatorname{argmax}} \left\{ p(\lambda) \frac{1}{(\sigma\sqrt{2\pi})^n} \exp\left(-\frac{\parallel \mathbf{y}-\lambda \parallel^2}{2\sigma^2}\right)\right\} \\ &= \underset{\lambda\in\Lambda_{\mathrm{s}}}{\operatorname{argmin}} \left\{ -\ln\left(p(\lambda)\right) + \frac{\parallel \mathbf{y}-\lambda \parallel^2}{2\sigma^2}\right\} \end{aligned} \tag{2}$$

III. A BOUND ON THE ERROR PROBABILITY UNDER OPTIMAL MAP DECODING

Using the optimal MAP decoder, we derive in the following theorem a union bound estimate on the sum codeword decoding error probability.

Theorem: III.1 *Consider a lattice design $(\Lambda, \mathcal{R})$, and a relay node computing a noiseless sum of K source codewords in a real-valued Gaussian multiple access channel using the optimal maximum a posteriori decoder. Then the union bound estimate of the probability of decoding error is*

$$\mathrm{P_{error}} \leq \frac{1}{2} \sum_{\lambda\in\Lambda_{\mathrm{s}}} \sum_{\hat{\lambda}\in\Lambda_{\mathrm{s}}\backslash\lambda} p(\lambda) \mathrm{erfc}\left(\sqrt{A} + \frac{B}{\sqrt{A}}\right) \tag{3}$$

where $A = \frac{d_{\min}^2}{8\sigma^2}$, $B = \frac{1}{4}\ln\left(\frac{p(\lambda)}{p(\hat{\lambda})}\right)$ *and* $d_{\min}$ *denotes the minimum distance of the coding lattice* Λ.

Proof The proof of our theorem is based on the pairwise error probability defined as the probability that the sum codeword λ has a larger MAP decoding metric in (2) than $\hat{\lambda}$ given that λ is transmitted. Its expression is formulated as follows

$$\begin{aligned}\Pr(\lambda \longrightarrow \hat{\lambda}) &= \Pr\left(-\ln\left(p(\hat{\lambda})\right) + \frac{\parallel \mathbf{y}-\hat{\lambda} \parallel^2}{2\sigma^2} < -\ln\left(p(\lambda)\right) + \frac{\parallel \mathbf{y}-\lambda \parallel^2}{2\sigma^2}\right) \\ &= \Pr\left(\ln\left(\frac{p(\lambda)}{p(\hat{\lambda})}\right) + \frac{\parallel \mathbf{y}-\hat{\lambda} \parallel^2}{2\sigma^2} - \frac{\parallel \mathbf{y}-\lambda \parallel^2}{2\sigma^2} < 0\right) \\ &= \Pr\left(2\sigma^2\ln\left(\frac{p(\lambda)}{p(\hat{\lambda})}\right) + \parallel \lambda-\hat{\lambda} \parallel^2 + 2\left\langle \lambda-\hat{\lambda}, \mathbf{z}\right\rangle < 0\right) \\ &= \Pr\left(\mathrm{G} < 0\right) = \mathrm{Q}\left(\frac{\mu_{\mathrm{G}}}{\sigma_{\mathrm{G}}}\right) \\ &= \mathrm{Q}\left(\frac{\parallel \lambda-\hat{\lambda} \parallel}{2\sigma} + \frac{\sigma}{\parallel \lambda-\hat{\lambda} \parallel}\ln\left(\frac{p(\lambda)}{p(\hat{\lambda})}\right)\right)\end{aligned}$$

where it is easy to prove that

$$\mathrm{G} = 2\sigma^2\ln\left(\frac{p(\lambda)}{p(\hat{\lambda})}\right) + \parallel \lambda-\hat{\lambda} \parallel^2 + 2 < \lambda-\hat{\lambda}, \mathbf{z} >$$

is a random Gaussian variable of mean μ_{G} and variance σ_{G}^2 given by:

$$\mu_{\mathrm{G}} = \| \lambda - \hat{\lambda} \|^2 + 2\sigma^2 \ln \left(\frac{p(\lambda)}{p(\hat{\lambda})} \right) \quad (4)$$

$$\sigma_{\mathrm{G}}^2 = 4\sigma^2 \| \lambda - \hat{\lambda} \|^2 \quad (5)$$

Using the union bound, we get,

$$\begin{aligned} \mathrm{P_{error}} &\leq \sum_{\lambda \in \Lambda_{\mathrm{s}}} p(\lambda) \sum_{\hat{\lambda} \in \Lambda_{\mathrm{s}} \setminus \lambda} \Pr(\lambda \longrightarrow \hat{\lambda}) \\ &\leq \sum_{\lambda \in \Lambda_{\mathrm{s}}} \sum_{\hat{\lambda} \in \Lambda_{\mathrm{s}} \setminus \lambda} p(\lambda) \mathrm{Q} \left(\frac{\| \lambda - \hat{\lambda} \|}{2\sigma} + \frac{\sigma}{\| \lambda - \hat{\lambda} \|} \ln \left(\frac{p(\lambda)}{p(\hat{\lambda})} \right) \right) \end{aligned}$$

Where Q(.) denotes the Q function. We can therefore, using the relation $\mathrm{Q}(x) = \frac{1}{2}\mathrm{erfc}(\frac{x}{\sqrt{2}})$, write:

$$\mathrm{P_{error}} \leq \frac{1}{2} \sum_{\lambda \in \Lambda_{\mathrm{s}}} \sum_{\hat{\lambda} \in \Lambda_{\mathrm{s}} \setminus \lambda} p(\lambda) \mathrm{erfc} \left(\frac{\| \lambda - \hat{\lambda} \|}{2\sqrt{2}\sigma} + \frac{\sigma}{\sqrt{2} \| \lambda - \hat{\lambda} \|} \ln \left(\frac{p(\lambda)}{p(\hat{\lambda})} \right) \right)$$

The last step to prove our theorem is based on two facts:

- $\| \lambda - \hat{\lambda} \| \geq d_{\min}$, for all $\lambda, \hat{\lambda} \in \Lambda_{\mathrm{s}}$. This inequality results from the linearity structure and the geometrical properties of the lattice Λ.
- The function $\mathrm{erfc}(x + \frac{\alpha}{x})$, $\alpha \in \mathbb{R}$ is a decreasing function with respect to x [10].

The proof follows then by considering A and B as defined above.

Given the derived upper bound, we propose a lattice design criterion as follows.

Proposition III.2 *Minimization of the error probability under MAP decoding requires to select lattice designs $(\Lambda, \mathcal{R})$ such that the minimum distance of the lattice Λ is maximized.*

Proof The upper bound on the error probability is a strictly decreasing function of A [10], thus a decreasing function of the minimum distance of the lattice Λ. Then in order to make the error probability small, the coding lattice Λ has to have a large minimum distance $d_{\min}$.

The construction of such good codes is out of the scope of this work. Even though, we point out that for lattices built using Construction A [11] over linear codes, this criterion requires to design linear codes with minimum euclidean weights.

IV. A NOVEL MAP DECODING ALGORITHM

Our objective in the following is to develop a practical decoding algorithm that allows to reliably find the optimal MAP estimate of the optimization problem in (2).

A first obvious approach consists in performing an exhaustive naive search over the sum codebook Λ_{s}. Using this approach, no assumptions on the sum codebook distribution is considered. The relay in this case, given the number of sources and the original codebook $\mathcal{C}$, derives the statistics of the sum codebook to compute the corresponding values of $p(\lambda)$ for all codewords $\lambda \in \Lambda_{\mathrm{s}}$, then, it exhaustively seeks the codeword which maximizes the decoding metric in (2). However, this method is not practical for two main reasons: its high complexity and the requirement of the knowledge of the instantaneous values of the probability distribution function for all sum codewords at the relay. In order to overcome these limitations to build a practical decoding algorithm, we analyze in the following the distribution of the sum codewords.

A. Discrete Gaussian Distribution of the Sum Codewords

The original codewords are drawn uniformally and independently from the uniform codebook $\mathcal{C}$, they are modeled by uniform random variables of zero-mean ($\mu_{\mathbf{x}} = 0$) and variance $\sigma_{\mathbf{x}}^2 = \frac{1}{n}\mathbb{E}\left(\| \mathbf{x}_i \|^2\right) \leq P$ for $i = 1, ..., K$. Consider now the sum codewords $\lambda = \sum_{i=1}^{K} \mathbf{x}_i$ obtained through the superposition of the vectors sent by the sources. Given the uniform distribution of the original codewords, The *Central Limit Theorem* states that λ is a random variable of mean $\mu_{\mathrm{s}} = K\mu_{\mathbf{x}} = 0$ and variance $\sigma_{\mathrm{s}}^2 = K\sigma_{\mathbf{x}}^2$. Particularly, for increasing number of sources K, the sum codewords converge to a normal distribution $\mathcal{N}\left(\mu_{\mathrm{s}}, \sigma_{\mathrm{s}}^2 \mathbf{I}_n\right)$. In order to be able to use this result to approximate the vectors λ by random Gaussian variables, we need in addition to take into consideration the fact that the sum codewords are discrete and correspond to lattice points. For this purpose we introduce the lattice Gaussian distribution. This tool arises in several problems in coding theory [12] and mathematics [13].

Let $f_{\sigma_{\mathrm{s}}}(\mathbf{x})$ denote the Gaussian distribution of variance σ_{s}^2 centered at the zero vector such that for $\sigma_{\mathrm{s}} > 0$ and all $\mathbf{x} \in \mathbb{R}^n$:

$$f_{\sigma_{\mathrm{s}}}(\mathbf{x}) = \frac{1}{\left(\sqrt{2\pi}\sigma_{\mathrm{s}}\right)^n} e^{-\frac{\|\mathbf{x}\|^2}{2\sigma_{\mathrm{s}}^2}} \quad (6)$$

Consider also the $\Lambda-$periodic function $f_{\sigma_{\mathrm{s}}}(\Lambda)$ defined by:

$$f_{\sigma_{\mathrm{s}}}(\Lambda) = \sum_{\lambda \in \Lambda} f_{\sigma_{\mathrm{s}}}(\lambda) = \frac{1}{\left(\sqrt{2\pi}\sigma_{\mathrm{s}}\right)^n} \sum_{\lambda \in \Lambda} e^{-\frac{\|\lambda\|^2}{2\sigma_{\mathrm{s}}^2}} \quad (7)$$

Then the sum codewords can be modeled by the discrete Gaussian distributions over Λ centered at the zero vector according to: $p(\lambda) = \frac{f_{\sigma_{\mathrm{s}}}(\lambda)}{f_{\sigma_{\mathrm{s}}}(\Lambda)}$. Illustrated examples in Fig.1 show that the discrete Gaussian distribution fits our settings. As a proof of concept, we will show by numerical results that this Gaussian model is well justified in the context of lattice network coding even for low number of sources $K \geq 2$.

B. Equivalent Novel MAP Decoding metric

Considering the Gaussian distribution on the sum codewords, equation (2) is equivalently written as:

$$\hat{\lambda}_{\mathrm{map}} = \underset{\lambda \in \Lambda_{\mathrm{s}}}{\mathrm{argmin}} \left\{ \ln\left(f_{\sigma_{\mathrm{s}}}(\Lambda)\right) + n \ln\left(\sigma_{\mathrm{s}}\sqrt{2\pi}\right) + \frac{1}{2\sigma_{\mathrm{s}}^2} \| \lambda \|^2 + \frac{1}{2\sigma^2} \| \mathbf{y} - \lambda \|^2 \right\}$$

The first and second terms in this optimization problem are independent of the variable λ, they can be disregarded in the optimization over λ. Then we define a new MAP decoding metric given by:

$$\hat{\lambda}_{\mathrm{map}} = \underset{\lambda \in \Lambda_{\mathrm{s}}}{\mathrm{argmin}} \left\{ \| \mathbf{y} - \lambda \|^2 + \beta^2 \| \lambda \|^2 \right\} \quad (8)$$

where $\beta = \frac{\sigma}{\sigma_{\mathrm{s}}}$. Using this new metric, we show in Proposition.IV.1 that MAP decoding reduces to solve for a closest vector problem.

Proposition IV.1 *The MAP decoding metric in (8) is equivalent to find the closest vector in the lattice Λ_{aug} of generator matrix $\mathbf{M}_{\mathrm{aug}} = [\mathbf{M}\ \beta\mathbf{M}]^t \in \mathbb{R}^{2n \times n}$ to the vector $\mathbf{y}_{\mathrm{aug}} = [\mathbf{y}\ \mathbf{0}_n]^t$ according to the following metric:*

$$\hat{\lambda}_{\mathrm{map}} = \underset{\substack{\mathbf{x}_{\mathrm{aug}} \in \Lambda_{\mathrm{aug}}/ \\ \mathbf{x}_{\mathrm{aug}} = \mathbf{M}_{\mathrm{aug}}\lambda}}{\mathrm{argmin}} \| \mathbf{y}_{\mathrm{aug}} - \mathbf{x}_{\mathrm{aug}} \|^2 \quad (9)$$

Proof The decoding metric in (8) can be written as:

$$\hat{\lambda}_{\text{map}} = \underset{\lambda\in\Lambda_{\text{s}}}{\text{argmin}} \left\{ \left\| \begin{bmatrix} \mathbf{y} \\ \mathbf{0}_n \end{bmatrix} - \begin{bmatrix} \lambda \\ \beta\lambda \end{bmatrix} \right\|^2 \right\} = \underset{\lambda\in\Lambda_{\text{s}}}{\text{argmin}} \parallel \mathbf{y}_{\text{aug}} - \mathbf{I}_{\text{aug}}\lambda \parallel^2 \quad (10)$$

where $\mathbf{I}_{\text{aug}} = [\mathbf{I}_n \ \beta\mathbf{I}_n]^t \in \mathbb{R}^{2n\times n}$ is a full rank matrix. On the other hand, given that the sum codewords belong to the fine lattice according to the shaping region $\mathcal{R}_{\text{s}}$, any codeword λ can be written in the form $\lambda = \mathbf{M}\mathbf{u}$ where $\mathbf{u} \in \mathcal{A}_{\text{s}} \subset \mathbb{Z}^n$ and $\mathcal{A}_{\text{s}}$ translates the shaping constraint imposed by $\mathcal{R}_{\text{s}}$ and can be deduced from the shaping boundaries limited by the transmission power constraint P. Consequently, the optimization problem in (10) is equivalent to solving

$$\hat{\lambda}_{\text{map}} = \underset{\mathbf{u}\in\mathcal{A}_{\text{s}}/\lambda=\mathbf{M}\mathbf{u}}{\text{argmin}} \parallel \mathbf{y}_{\text{aug}} - \mathbf{I}_{\text{aug}}\mathbf{M}\mathbf{u} \parallel^2 = \underset{\mathbf{u}\in\mathcal{A}_{\text{s}}/\lambda=\mathbf{M}\mathbf{u}}{\text{argmin}} \parallel \mathbf{y}_{\text{aug}} - \mathbf{M}_{\text{aug}}\mathbf{u} \parallel^2 \quad (11)$$

$\mathbf{M}_{\text{aug}}$ is a full rank matrix and $\mathbf{u}$ is an integer vector, then solving (11) consists in finding the closest vector $\mathbf{x}_{\text{aug}} = \mathbf{M}_{\text{aug}}\mathbf{u}$ to $\mathbf{y}_{\text{aug}}$ in the $n-$dimensional lattice Λ_{aug} of a generated matrix $\mathbf{M}_{\text{aug}}$. After finding the optimal integer vector $\mathbf{u}_{\text{opt}}$ that minimizes the metric in (11), the optimal MAP estimate is deduced by $\hat{\lambda}_{\text{map}} = \mathbf{M}\mathbf{u}_{\text{opt}}$.

In practice, the sphere decoder can be used to solve the closest vector problem. We propose in this work a modified version of this algorithm to take into account the shaping constraint.

Remark: The MAP decoding metric in (8) involves two terms each one of them is given by an Euclidean distance. When the first term is dominant, which is the case when $\beta^2 = \frac{\sigma^2}{\sigma_{\text{s}}^2} = \frac{\sigma^2}{K\sigma_{\mathbf{x}}^2} \ll 1$, the MAP decoding rule reduces to minimum distance decoding which is equivalent to ML decoding in this case. Given that $\sigma_{\mathbf{x}}^2$ depends on the power constraint P, we deduce that this case of figure is likely to happen either at high signal to noise ratio or when $K\sigma_{\mathbf{x}}^2$ is sufficiently higher than the noise variance σ^2. We expect then that the MAP decoding and the conventional minimum distance decoder achieve similar performance at high SNR range. Adversely, at the low and moderate SNR regime and when the product $K\sigma_{\mathbf{x}}^2$ is small, the second term in the decoding metric applies an incremental constraint that considers the non-uniform distribution of the sum codewords in Λ_{s} which is not taken into account under the conventional decoder. In this case, we expect that the MAP decoder outperforms the minimum distance decoding-based one.

We provide in the following proposition an equivalent formulation of the MAP decoding metric related to perform MMSE-GDFE preprocessing followed by minimum distance decoding.

Proposition IV.2 *[Equivalence between MAP decoding and MMSE-GDFE preprocessed lattice decoding] The MAP decoding metric in (8) is equivalent to MMSE-GDFE preprocessed minimum Euclidean distance decoding according to the metric:*

$$\hat{\lambda}_{\text{map}} = \underset{\lambda\in\Lambda_{\text{s}}}{\text{argmin}} \parallel \mathbf{F}\mathbf{y} - \mathbf{B}\lambda \parallel^2 \quad (12)$$

where $\mathbf{F} \in \mathbb{R}^{n\times n}$ and $\mathbf{B} \in \mathbb{R}^{n\times n}$ denote respectively the forward and backward filters of the MMSE-GDFE preprocessing for the channel $\mathbf{y} = \lambda + \mathbf{z}$ given in (1) such that $\mathbf{B}^t\mathbf{B} = (1+\beta^2)\,\mathbf{I}_n$ and $\mathbf{F}^t\mathbf{B} = \mathbf{I}_n$.

Proof Let $N(\lambda)$ denote the metric we aim to minimize in (8), we have the following:

$$\begin{aligned} N(\lambda) &= \parallel \mathbf{y} - \lambda \parallel^2 + \beta^2 \parallel \lambda \parallel^2 \\ &= \mathbf{y}^t\mathbf{y} - 2\mathbf{y}^t\lambda + \lambda^t\lambda + \beta^2\lambda^t\lambda \\ &= (1+\beta^2)\,\lambda^t\lambda + \mathbf{y}^t\mathbf{y} - 2\mathbf{y}^t\lambda \\ &= \lambda^t\mathbf{B}^t\mathbf{B}\lambda + \mathbf{y}^t\mathbf{y} - 2\mathbf{y}^t\mathbf{F}^t\mathbf{B}\lambda \\ &= \underbrace{\lambda^t\mathbf{B}^t\mathbf{B}\lambda + \mathbf{y}^t\mathbf{F}^t\mathbf{F}\mathbf{y} - 2\mathbf{y}^t\mathbf{F}^t\mathbf{B}\lambda}_{\|\mathbf{F}\mathbf{y}-\mathbf{B}\lambda\|^2} + \underbrace{\mathbf{y}^t\left(\mathbf{I}_n - \mathbf{F}^t\mathbf{F}\right)\mathbf{y}}_{\Gamma(\mathbf{y})} \end{aligned}$$

where $\mathbf{F} \in \mathbb{R}^{n\times n}$ and $\mathbf{B} \in \mathbb{R}^{n\times n}$ are chosen such that: $\mathbf{B}^t\mathbf{B} = (1+\beta^2)\,\mathbf{I}_n$ and $\mathbf{F}^t\mathbf{B} = \mathbf{I}_n$. Given that $\Gamma(\mathbf{y}) > 0$ and independent of λ, minimization of $N(\lambda)$ is equivalent to minimize $\parallel \mathbf{F}\mathbf{y} - \mathbf{B}\lambda \parallel^2$. The last piece to our proof is to show that the matrices $\mathbf{F}$ and $\mathbf{B}$ correspond to the filters of the MMSE-GDFE preprocessing in the system $\mathbf{y} = \lambda + \mathbf{z}$ of input λ and AWGN $\mathbf{z}$. This proof is ommited for space limitation.

In order to find the MAP estimate according to the decoding metric in (12), the receiver, given the channel output, first performs MMSE-GDFE preprocessing, then performs minimum Euclidean distance decoding to find the nearest point to $\mathbf{F}\mathbf{y}$ in the lattice of generator matrix $\mathbf{B}\mathbf{M}$ according to the shaping constraint imposed by the subset Λ.

V. NUMERICAL RESULTS

We provide in this section numerical results obtained through Monte-Carlo simulations and evaluating the codeword error rate at the relay for the conventional minimum distance decoder and the proposed MAP decoding algorithm implementing a modified sphere decoder. In addition, in order to validate the Gaussianity law assumption we considered to derive our MAP decoding metric, we include the naive exhaustive search to solve (2). We studied in our analysis two lattice examples as described below.

Example 1: 2-Dimensional lattice ($n = 2$): for this first example we choose a 2-dimensional lattice Λ of a generator matrix $\mathbf{M} = \begin{bmatrix} 2 & 3 \\ 3 & -1 \end{bmatrix}$. In addition, we consider a spherical shaping region given by the power constraint P, and study the cases of $K = 2$ and $K = 5$ which correspond to the statistical distributions depicted in Fig.1. The shaping constraint in this case is given by $P = \sigma_{\mathbf{x}}^2 = 6.5$. Given the number of sources and the power constraint, we calculate for each case the bounds requirements to be considered in the decoding process using the sphere decoder. Numerical results concerning the first case, depicted in Fig.2, show that our proposed algorithm achieves almost identical performance as the exhaustive search, which confirms the effectiveness of our metric as well as the validity of the Gaussianity law assumption considered to model the sum-codewords even for the case of low number of sources K. Moreover, plotted curves

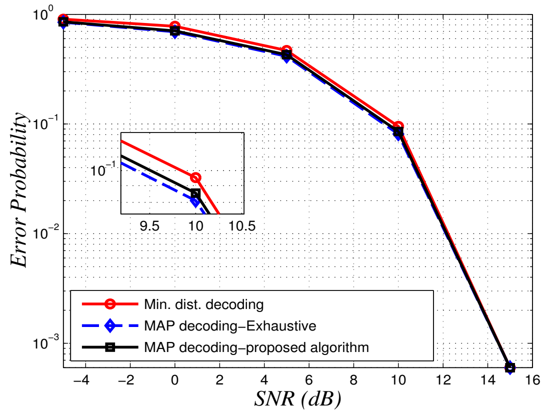

Fig. 2. Error performance for the case $n = 2, K = 2, P = 6.5$.

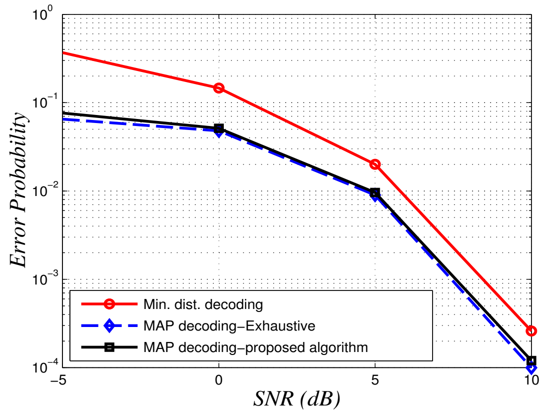

Fig. 4. Error performance for $n = 4, K = 2, P = 1$.

show that the MAP decoder outperforms the conventional minimum distance decoding (Min. dist. decoding). The gain for this 2-dimensional lattice case is not huge, and is limited to 0.5dB for a codeword error rate of 10^{-1}. Results for the case of $K = 5$ plotted in Fig.3 confirm the previous findings and show that the performance gap between the MAP and the Minimum distance decoder is also not high. Common to these two settings is the high value of $K\sigma_{\mathbf{x}}^2$, which joins our previous analysis.

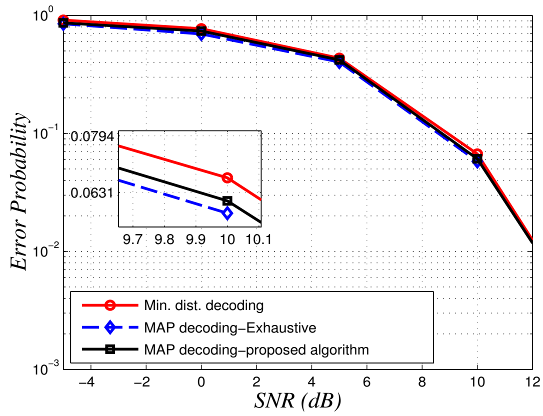

Fig. 3. Error performance for $n = 2, K = 5, P = 6.5$.

Example 2: 4-Dimensional lattice $(n = 4)$*:* in this second example we consider the 4-dimensional integer lattice Λ of a generator matrix the identity $\mathbf{I}_4$ together with a cubic shaping region according to $P = 1$. The aim of considering this example is to analyze the performance of the MAP decoder when the lattice dimension increases. Simulation results depicted in Fig.4 show that our proposed MAP algorithm allows to achieve a gain of 1dB at a codeword error rate of 10^{-3} over the minimum distance decoder while keeping a small gap to the exhaustive search. This case shows the merit of applying the MAP decoding in settings where the product $K\sigma_{\mathbf{x}}^2$ is small. In addition, we notice that the gap between the MAP decoder and the conventional one is independent of the lattice dimension, it rather increases in settings involving small $K\sigma_{\mathbf{x}}^2$.

VI. Conclusion

In this work we studied optimal MAP decoding for lattice-based physical-layer network coding in the Gaussian multiple access relay channel. We proposed an analytical upper bound on the codeword error probability and developed a novel decoding algorithm inspired by the sphere decoder. Our simulation results show the effectiveness of the proposed algorithm and confirm the merit of applying the MAP criterion in particular for high dimensional lattices. We explore inn future works the capacity limits of the optimal MAP decoder.

References

[1] S Zhang. Hot topic: Physical-layer network coding. In *Proceedings of ACM Mobicom*, pages 358–365, 2006.
[2] B. Nazer and M. Gastpar. Compute-and-forward: Harnessing interference with structured codes. In *Proceedings of ISIT*, pages 772 –776, July 2008.
[3] C. Feng, D. Silva, and F.R. Kschischang. An algebraic approach to physical-layer network coding. In *Proceedings of ISIT*, pages 1017 –1021, June 2010.
[4] A. Mejri and G. Rekaya. Practical physical layer network coding in multi-sources relay channels via the compute-and-forward. In *Proceedings of WCNC*, pages 166–171, April 2013.
[5] A. Mejri and G. Rekaya. Bidirectional relaying via network coding: Design algorithm and performance evaluation. In *Proceedings of ICT*, pages 1–5, May 2013.
[6] A. Mejri, G. Rekaya, and J. C Belfiore. Lattice decoding for the compute-and-forward protocol. In *Proceedings of International Conference on Communications and Networking*, pages 1–8, March 2012.
[7] J. C Belfiore. Lattice codes for the compute-and-forward protocol: The flatness factor. In *Proceedings of ITW*, pages 1–4, 2011.
[8] M.P. Wilson, K. Narayanan, H.D. Pfister, and A. Sprintson. Joint physical layer coding and network coding for bidirectional relaying. *IEEE Trans. on IT*, 56(11):5641 –5654, November 2010.
[9] P. Popovski and H. Yomo. The anti-packets can increase the achievable throughput of a wireless multi-hop network. In *Proceedings of ICC*, pages 3885–3890, 2006.
[10] F. Behnamfar, F. Alajaji, and T. Linder. Performance analysis of map decoded space-time orthogonal block codes for non-uniform sources. In *Proceedings of ITW*, pages 46–49, March.
[11] U. Erez, S. Litsyn, and R. Zamir. Lattices which are good for (almost) everything. In *Proceedings of ITW*, pages 271 – 274, March 2003.
[12] G.D. Forney, M.D. Trott, and S-Y. Chung. Sphere-bound-achieving coset codes and multilevel coset codes. *IEE Trans. on IT*, 46(3):820–850, 2000.
[13] W. Banaszczyk. New bounds in some transference theorems in the geometry of numbers. *Math. Ann.*, 296:625–635, 1993.
[14] B. Nazer and M. Gastpar. Reliable physical layer network coding. *IEEE Trans. on IT*, 99(3):438 –460, March 2011.
[15] U. Fincke and M. Pohst. Improved Methods for Calculating Vectors of Short Length in a Lattice, Including a Complexity Analysis. *Mathematics of Computation*, 44(170):463–471, 1985.