Bidirectional Relaying Via Network Coding: Design Algorithm and Performance Evaluation

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***Abstract*—Research works shed considerable light on the merit of the Physical Layer Network Coding in the Two-Way Relay Channel using lattice codes. This potential is proved under a capacity achieving perspective. However, it is not completely understood if this promised gain is attainable in practical settings. We try in this work to answer to this issue by investigating two network coding strategies: the Compute-and-Forward and the Analog Network Coding. We analyze end-to-end communication using these methods and evaluate their performance in terms of error rate and exchange rate using a low-complexity lattice encoding scheme.**

***Index Terms*—Two-way relay channel, Compute-and-Forward, nested lattice codes.**

I. Introduction

We consider communication over the wireless Two-Way Relay Channel (TWRC) where two end nodes exchange messages with the help of a relay node, and consider the use of lattice codes for channel coding [1].

Using the traditional *Store-and-Forward* (SF) strategy, interference due to the superposition of source signals at the relay is avoided by transmission time scheduling. Data exchange lasts therefore four time slots (TSs).

More efficient transmission can be realized via Network Coding [2]. In this framework, interference is again avoided, however, instead of forwarding a whole copy of the original data as the case of the SF method, the relay decodes signals separately, combines them into a function then forwards the computed output to end nodes. Information exchange takes thus only 3TSs and higher network throughput can be attained.

Under a newer coding perspective termed *Physical Layer Network Coding* (PNC) [3], interference is exploited and used to design more reliable network codes. The aim of this technique is to allow simultaneous transmission from different devices in a network, and enable intermediate nodes to decode and forward a function of superposed signals with no prior decoding of each one of them separately. In the TWRC, this makes information exchange last only 2TSs and results in higher data rates.

PNC can be performed joint to physical layer techniques such as the modulation [4],[5]-[6] and channel coding [7],[8]. In this work we are interested in the second category. We consider in particular lattice-based channel coding schemes and study two PNC strategies: the *Compute-and-Forward* (CF) and the *Analog Network Coding* (ANC). These strategies have been initially studied in [7] and [8] for the Gaussian and fading TWRC. Research works show the promise of the CF and prove its superiority over the ANC for both channel cases only from a theoretical capacity achieving perspective. But what about the practice? is the promised potential of the CF attainable in practical end-to-end communication scenarios? and what would be the difference between the two channels? We try in this work to answer to these issues. We analyze a practical bidirectional transmission in the Gaussian and fading channels and evaluate numerical performance of both strategies at end nodes in terms of exchange rate and error rate using a low-complexity lattice encoding scheme. Our contributions are the following:

• We show that for the Gaussian channel the CF achieves lower error probability than the ANC. The former offers a gain of 3.75dB over the latter at a message error rate of 10^{-2}.

• We review the design criterion for optimal network code vector for the CF in the fading channel. Resulting optimization problem was only highlighted in [7] but no explicit methods to solve it have been proposed so far. We propose in this work a search algorithm for optimal network code vector selection and carry out experimental studies to confirm its efficiency.

• We show that unlike the Gaussian channel case, in the fading channel the CF achieves almost same error rate performance as the ANC. Superiority of the CF is only reported in terms of the achievable exchange rate.

The remaining content of this work is organized as follows: basic lattice definitions are introduced in section II. The TWRC model and the encoding scheme are presented in section III. Description of the CF and the ANC as well as their performance evaluation and comparison for the Gaussian and fading channels are addressed respectively in sections IV and V. The work is ended with a concluding section.

II. Preliminaries: lattice definitions

An n-dimensional ***lattice*** Λ is a set of points of $\mathbb{R}^n$ given by $\Lambda = \{\mathbf{x} = \mathbf{M}\mathbf{s}, \mathbf{s} \in \mathbb{Z}^n\}$ where $\mathbf{M}$ is called a generator matrix of the lattice. The main characteristic of Λ is *linearity*, i.e. for any $a, b \in \mathbb{Z}$ and $\mathbf{x}, \mathbf{y} \in \Lambda$, $a\mathbf{x} + b\mathbf{y} \in \Lambda$.

A ***lattice quantizer*** Q_Λ is the mapping that takes a real vector $\mathbf{x}$ to the nearest point in Λ in Euclidean distance as $Q_\Lambda(\mathbf{x}) = \text{argmin}_{\lambda\in\Lambda} \parallel \mathbf{x} - \lambda \parallel$. The set of points that quantize to a given lattice point is called the ***Voronoi Region***. The ***fundamental Voronoi Region*** $\mathcal{V}_\Lambda$ of a lattice Λ corresponds to the voronoi region of the zero vector.

The $\mathbf{mod} - \mathbf{\Lambda}$ operation returns the quantization error with respect to Λ. For $\mathbf{x} \in \mathbb{R}^n$: $[\mathbf{x}] \bmod \Lambda = \mathbf{x} - Q_\Lambda(\mathbf{x})$.

A ***nested lattice code*** Λ is the set of all points of a lattice Λ_{F} (termed the *Fine* lattice) that fall within the fundamental

Voronoi Region of a lattice Λ_{C} (termed the *Coarse* lattice) as: $\Lambda = \{\lambda = [\lambda_{\mathrm{F}}] \operatorname{mod}\Lambda_{\mathrm{C}}, \lambda_{\mathrm{F}} \in \Lambda_{\mathrm{F}}\}$

III. SYSTEM MODEL AND ASSUMPTIONS

We consider a TWRC composed of two nodes N_1 and N_2 that wish to exchange their messages with the help of a relay node R. All nodes are equipped with a single antenna and operate in half duplex mode. Messages exchange is made through two orthogonal phases as follows.

A. Uplink phase

During this phase, source node N_1 (resp. N_2) delivers a message $\mathbf{w}_1$ (resp. $\mathbf{w}_2$) of length k drawn i.i.d from a prime size field $\mathbb{F}_p$ according to a uniform distribution. Sources encode their messages onto $n-$dimensional lattice codewords $\mathbf{x}_i$ satisfying a symmetric power constraint given by $\frac{1}{n} \parallel \mathbf{x}_i \parallel^2 \leq P,\ i = 1, 2$.

We consider in this work a particular class of lattice codes called *nested lattice codes*. They are proved to achieve the capacity of the AWGN channel [1] and to attain high rates in the case of the Multiple Access Channels [7] when coupled with physical layer network coding. A main objective of this work is to analyze the performance of these codes in practical settings considering finite-length codes.

According to this coding scheme, nodes N_1 and N_2 use the same one-to-one mapping ϕ to construct their codewords from a same nested lattice code Λ such that:

$$\begin{aligned} \phi :\ \mathbb{F}_p &\longrightarrow \Lambda = \{\mathcal{V}_{\mathrm{C}} \cap \Lambda_{\mathrm{F}}\} \\ \mathbf{w}_i &\longmapsto \mathbf{x}_i,\ i = 1, 2 \end{aligned} \tag{1}$$

The considered fine lattice Λ_{F} is assumed to be a good channel code [1] from which are carved the codewords, and the coarse lattice serves as a shaping lattice to ensure the transmission power constraint P.

In practice, a nested lattice code can be constructed from linear codes (or LDPC codes) over the field $\mathbb{F}_p$. The fine lattice is generated by applying the Construction A to the underlying linear code [1] and the coarse lattice can just be $\Lambda_{\mathrm{C}} = p\mathbb{Z}^n$. In our performance evaluation we consider a two-dimensional lattice over $\mathbb{Z}_{11}$. Our considered coarse lattice is $\Lambda_{\mathrm{C}} = 11\mathbb{Z}^2$ and our fine lattice is generated using the linear code of generator matrix $\mathbf{G} = [2\ 3]$.

Encoded vectors are afterwards transmitted simultaneously to the relay node. For ease of presentation, only real fading channels are considered. Obtained results can be extended to the complex-valued channel case using the model of [9]. Received signal at the relay can then be written as,

$$\mathbf{y}_R = h_1\mathbf{x}_1 + h_2\mathbf{x}_2 + \mathbf{z}_R \tag{2}$$

where h_1, h_2 denote the real channel coefficients generated i.i.d according to a normal distribution $\mathcal{N}(0, 1)$ and $\mathbf{z}_R \in \mathbb{R}^n$ denotes the zero-mean AWGN of variance σ^2. We assume that a perfect Channel State Information is available only at the receiver and define the Signal to Noise Ratio (SNR) as $\rho = \frac{P}{\sigma^2}$. At the end of this phase, the relay decodes from $\mathbf{y}_R$ a function of original codewords $\mathbf{x}_R = f(\mathbf{x}_1, \mathbf{x}_2)$ satisfying the same power constraint P as the source nodes and broadcasts it to N_1 and N_2.

B. Downlink Phase

Received signals at end nodes can be written as:

$$\mathbf{y}_i = h_i\mathbf{x}_R + \mathbf{z}_i,\ i = 1, 2 \tag{3}$$

where $\mathbf{z}_i$ denotes a zero-mean AWGN of variance $\sigma_i^2 = \sigma^2$ and channel gains are assumed to be identical to the uplink phase channel coefficients. From $\mathbf{y}_i$, node $N_i (i = 1, 2)$ uses its side information to get an estimate of the desired message $\mathbf{w}_j (j = 2, 1)$.

In order to evaluate the end-to-end performance of the studied PNC strategies, we consider two relevant parameters: the *Sum Message Error Rate* (SMER) and the *exchange rate* [4].

The SMER is defined as the sum of the message error rate at the two sources and is obtained by the error probability:

$$\mathrm{P_e} \triangleq \Pr(\hat{\mathbf{w}}_1 \neq \mathbf{w}_1) + \Pr(\hat{\mathbf{w}}_2 \neq \mathbf{w}_2) \tag{4}$$

The exchange rate is defined as the achievable rate per transmitter per channel use (a channel use signifies the use of the uplink and downlink phases). It is equal to the minimum rate between the rate achievable during the uplink $\mathcal{R}_{N_i \to R}$ and that achievable during the downlink phase $\mathcal{R}_{R \to N_j}$ as:

$$\mathcal{R}_{\mathrm{ex}} = \mathcal{R}_{N_i \to N_j} = min\left(\mathcal{R}_{N_i \to R}, \mathcal{R}_{R \to N_j}\right) \tag{5}$$

The upper bound on the exchange rate, obtained by the cut set bound, is included for comparison and given by:

$$\mathcal{R}_{\mathrm{ex,UB}} = \min_{m=1,2} \left\{ \frac{1}{2} \log\left(1 + h_m^2 \frac{P}{\sigma^2}\right)\right\} \tag{6}$$

IV. GAUSSIAN TWRC

In this first channel case, we assume that $h_1 = h_2 = 1$. The received signal at the relay is then equal to

$$\mathbf{y}_R = \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{z}_R \tag{7}$$

A. Compute-and-Forward

1) Processing at the relay: the relay attempts to decode a noiseless sum of the original codewords in the form:

$$\mathbf{x}_R = [\mathbf{x}_1 + \mathbf{x}_2] \operatorname{mod}\Lambda_{\mathrm{C}} \tag{8}$$

Thanks to the linearity of the lattice code, this desired sum is also a codeword from the same nested lattice code Λ and meets the same power constraint as original codewords.

The computation in the CF protocol is based on two operations: i) *scaling of the channel output* according to the *minimum mean square error* (MMSE) criterion and ii) *decoding desired combination using Minimum Distance Decoding*. The objective of the MMSE scaling step is to decrease the variance of the effective noise and consequently obtain higher rates. Authors in [8] prove that achievable rates are equal to

$$\mathcal{R}_{N_i \to R,\mathrm{CF}} = \frac{1}{2} \log\left(\frac{1}{2} + \frac{P}{\sigma^2}\right) \tag{9}$$

In the case of the AWGN channel case, the MMSE scaling factor is equal to $\alpha = \frac{2\rho}{1+2\rho}$. Different steps for the decoding operation are the following:

i) Scaling of the channel output as $\tilde{\mathbf{y}}_R = \alpha \mathbf{y}_R$.
ii) Quantization to the fine lattice: $Q_{\Lambda_F}(\tilde{\mathbf{y}}_R)$.
iii) Performing modulo operation with respect to the coarse lattice to get $\hat{\mathbf{x}}_R = [Q_{\Lambda_F}(\tilde{\mathbf{y}}_R)] \bmod \Lambda_C$.

2) Processing at end nodes: Node N_i receives $\mathbf{y}_i = \mathbf{x}_R + z_i$ $(i = 1, 2)$ and makes the following processing:

i) Decode $\hat{\mathbf{x}}_R = \operatorname{argmin}_{\lambda \in \Lambda} \| \mathbf{y}_i - \lambda \|^2$ under a Maximum Likelihood (ML) Decoding .
ii) Map $\hat{\mathbf{x}}_R$ to the finite field to get $\mathbf{u} = \phi^{-1}(\hat{\mathbf{x}}_R)$. Since $\hat{\mathbf{x}}_R$ is a nested lattice codeword, its mapping to the finite field generates an estimate of the sum of the finite field messages, that is $\mathbf{u} = \mathbf{w}_1 \oplus \mathbf{w}_2$.
iii) Substract message side information: $\mathbf{u} \ominus \mathbf{w}_1 = \hat{\mathbf{w}}_2$.

Where $\oplus$ and $\ominus$ denote respectively the addition and substraction operations over $\mathbb{F}_p$.

Given ML decoding at the downlink channels, corresponding achievable rate is equal to the channel capacity. The exchange rate is consequently equal to:

$$\mathcal{R}_{\text{ex,CF}} = \frac{1}{2} \log \left(\frac{1}{2} + \frac{P}{\sigma^2} \right) \tag{10}$$

Remark: End-to-end decoding errors depend both on the correctness of the decoding of the sum of codewords at the relay, and the decoding of $\hat{\mathbf{x}}_R$ in the downlink phase at end nodes. The first error type may result from the suboptimality of the minimum distance decoding compared to the ML decoding [10] . The second case is likely to happen since the two downlink channels are perturbed by different noises $\mathbf{z}_1$ and $\mathbf{z}_2$. Then, although end nodes use a ML decoding, they may not decode the same combination estimated at the relay's level.

Remark: The side information at end nodes is used at the messages' level. However, it can be also made at the codewords' level by inverting steps ii) and iii) as follows: ii) Substract codeword side information $\mathbf{x}_i$ from $\hat{\mathbf{x}}_R$ as $\hat{\mathbf{x}}_j = [\hat{\mathbf{x}}_R - \mathbf{x}_i] \bmod \Lambda_C$. iii) Map to the finite field $\hat{\mathbf{w}}_j = \phi^{-1}(\hat{\mathbf{x}}_j)$. The inversion of the two steps does not impact the decoding error since the mapping to the finite field does not change the correctness of the decoding.

B. Analog Network Coding

1) Processing at the relay: Using the ANC [7], the relay amplifies its input as $\mathbf{x}_R = \beta \mathbf{y}_R$, where β is selected such that the power constraint P is satisfied. Its value for the Gaussian TWRC is equal to: $\beta = \sqrt{\frac{\rho}{1+2\rho}}$.

2) Processing at end nodes: End node N_i receives: $\mathbf{y}_i = \beta \mathbf{y}_R + \mathbf{z}_i$ and performs the following steps:

i) Substract side information as $\tilde{\mathbf{y}}_i = \beta \mathbf{x}_j + \beta \mathbf{z}_R + \mathbf{z}_i$.
ii) Decode $\hat{\mathbf{x}}_j = \operatorname{argmin}_{\lambda \in \Lambda} \| \tilde{\mathbf{y}}_i - \beta\lambda \|^2$.
iii) Map to the finite field to get $\hat{\mathbf{w}}_j = \phi^{-1}(\hat{\mathbf{x}}_j)$.

The achievable exchange rate using the ANC is equal to:

$$\mathcal{R}_{\text{ex,ANC}} = \frac{1}{2} \log \left(1 + \frac{\rho^2}{1 + 3\rho} \right) \tag{11}$$

Where $\text{SNR}_{\text{eq}} = \frac{\rho^2}{1+3\rho}$ is the effective SNR at end nodes after susbtraction of codeword side information.

C. Numerical results

In this subsection we provide numerical results obtained through Monte-Carlo simulations using the nested lattice code described in section III. Starting with the exchange rate depicted in Fig.1, our numerical results confirm the optimality of the CF over the ANC. While the former reaches the upper bound at high SNR, the latter represents a constant gap to the channel capacity. This suboptimality is explained by the fact that the ANC amplifies the noise at the relay's level, however, the CF allows to eliminate it. As far as the Sum-MER is concerned, our results plotted in Fig.2 show that the CF allows to achieve lower error rate than the ANC. It presents a gain of 3.75dB at a MER=10^{-2}. Our numerical results bridge theory with practice and confirm thus the superiority of the CF over the ANC in practical settings.

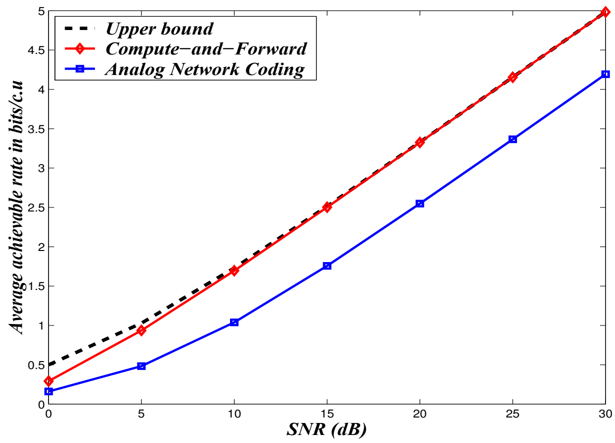

Fig. 1. Average exchange rate for the Gaussian TWRC.

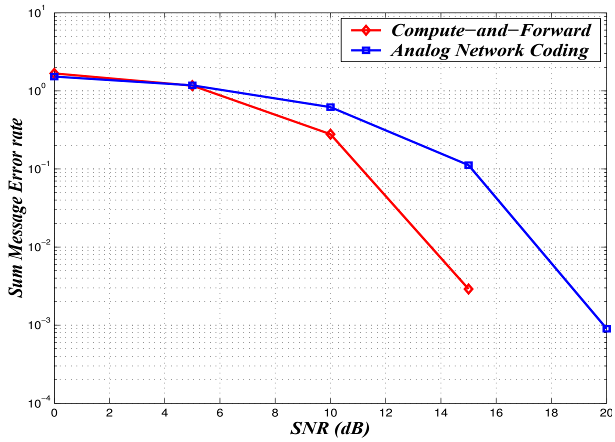

Fig. 2. Message Error rate for the Gaussian TWRC.

V. Real Fading TWRC

In this section the channel model of (2) is considered.

A. Compute-and-Forward

1) Processing at the relay: the relay aims to decode an *integer linear combination* of the original codewords in the form:

$$\mathbf{x}_R = [a_1\mathbf{x}_1 + a_2\mathbf{x}_2] \bmod \Lambda_C \tag{12}$$

where $a_1, a_2 \in \mathbb{Z}$ are freely selected by the relay to quantize the real channel coefficients in order to approach the channel output to a lattice codeword. This approximation is made by the MMSE scaling of the channel output. In the fading channel case, the MMSE scaling factor is given $\alpha = \frac{\rho\langle \mathbf{h}^t, \mathbf{a}\rangle}{1+\rho\|\mathbf{h}\|^2}$. Then the relay performs the same decoding steps detailed in the previous section.

Authors in [7] prove that, for given vectors $\mathbf{h} = [h_1\ h_2]^t$ and $\mathbf{a} = [a_1\ a_2]^2$ the achievable comptatiton rate is equal to:

$$\mathcal{R}_{N_i \to R} = \frac{1}{2}\log^+\left(\left(\|\mathbf{a}\|^2 - \frac{\rho \mid \mathbf{h}^t\mathbf{a} \mid^2}{1+\rho \parallel \mathbf{h} \parallel^2}\right)^{-1}\right) \tag{13}$$

where $\log^+(x) = max(\log(x), 0)$.
In order to optimally select integer network code vector $\mathbf{a}$, the maximization of the computation rate in (13) is proposed as a design criterion [11]. Consequently, the optimal vector is a solution of the minimization problem given by:

$$\mathbf{a}_{\text{opt}} = \underset{\mathbf{a}\neq\mathbf{0}}{\operatorname{argmin}}\left\{\mathbf{a}^t\mathbf{G}\mathbf{a}\right\} \tag{14}$$

where $\mathbf{G} = \mathbf{I} - \frac{\rho}{1+\rho\|\mathbf{h}\|^2}\mathbf{h}\mathbf{h}^t$. $\mathbf{a}_{\text{opt}}$ corresponds to the shortest vector of the lattice $\Lambda_{\mathbf{G}}$ of Gram matrix $\mathbf{G}$ and can be found in practice using the Fincke-Pohst algorithm [12].

2) Processing at end nodes: Upon reception of $\mathbf{y}_i$, node N_i makes the following steps:

i) Decode $\hat{\mathbf{x}}_R = \operatorname{argmin}_{\lambda\in\Lambda} \parallel \mathbf{y}_i - h_i\lambda \parallel^2$.
ii) Map to the finite field: $\mathbf{u} = \phi^{-1}(\hat{\mathbf{x}}_R) = q_1\mathbf{w}_1 \oplus q_2\mathbf{w}_2$. Coefficients q_i are given by $q_i = [a_i] \bmod p$.
iii) Substract side information: $\mathbf{n}_i = \mathbf{u} \ominus q_i\mathbf{w}_i = q_j\mathbf{w}_j$.
iv) Invert by q_j: $\hat{\mathbf{w}}_j = \frac{\mathbf{n}_i}{q_j}$.

3) Condition for decoding at end nodes: : in order to enable the end nodes recover desired messages, the finite field coefficients q_1 and q_2 should be non zero, meaning that the network code vector should satisfy: $[a_1] \bmod p \neq 0$ and $[a_2] \bmod p \neq 0$ at the same time. However, the optimization problem in (14) rejects only the values in the form $[a_1\ 0]^t$ and $[0\ a_2]^t$ which are not sufficient to guarantee recovering both messages at the two nodes. We formulate in the following lemma the optimization problem taking the non-zero condition over the finite field into account.

Lemma V.1 *For the fading TWRC using the Compute-and-Forward, optimal network code coefficient vector is a solution to the minimization problem:*

$$\mathbf{a}_{\text{opt}} = \underset{[a_1]\bmod p\neq 0, [a_2]\bmod p \neq 0}{\operatorname{argmin}}\left\{\mathbf{a}^t\mathbf{G}\mathbf{a}\right\} \tag{15}$$

It corresponds to a shortest vector in the lattice $\Lambda_{\mathbf{G}}$ of Gram matrix $\mathbf{G}$ having non-zero entries modulo the field size p.

If the network code vector satisfies this condition, the exchange rate for the CF equals to:

$$\mathcal{R}_{\text{ex,CF}} = \frac{1}{2}\log^+\left(\left(\|\mathbf{a}_{\text{opt}}\|^2 - \frac{\rho \mid \mathbf{h}^t\mathbf{a}_{\text{opt}} \mid^2}{1+\rho \parallel \mathbf{h} \parallel^2}\right)^{-1}\right) \tag{16}$$

Otherwise, it is equal to 0.

Inspite being extremely important, this non-zero constraint is not well studied in literature. It was only mentioned in [7]. How to solve it in practice and what is its impact on the practical end-to-end performance in the TWRC are not totally understood. A main contribution of this work is to answer to these two issues. We propose a search method to solve (15) based on a modification of the Fincke-Pohst algorithm. The idea is to select among the lattice shortest vectors the one which satisfies the non zero condition.

Let $\mathbf{G} = \mathbf{R}^t\mathbf{R}$ be the Cholesky decomposition of the definite positive matrix $\mathbf{G}$ where $\mathbf{R}$ is an upper triangular real matrix. The quadratic form $Q(\mathbf{a}) = \mathbf{a}^t\mathbf{G}\mathbf{a}$ can then be transformed to:

$$Q(\mathbf{a}) = \parallel \mathbf{R}\mathbf{a} \parallel^2 = u_{11}^2\left(a_1 + u_{12}a_2\right)^2 + u_{22}a_2^2 \tag{17}$$

where $u_{ii} = R_{ii}^2, i = 1, 2$ and $u_{12} = \frac{R_{12}}{R_{11}}$. The obvious method to search the optimal vector is to make an exhaustive search over $\mathbb{Z}^2$ and find the vector minimizing $Q(\mathbf{a})$ and satisfying non-zero condition. However, this search leads to an increasing complexity specially when the SNR is high. The idea is to reduce the search space to the sphere of radius $C > 0$ such that $Q(\mathbf{a}) \leq C$. Satisfying this upper bound leads to the following boundaries on the coefficients a_1 and a_2 as:

$$-\sqrt{\frac{C}{u_{22}}} \leq a_2 \leq \sqrt{\frac{C}{u_{22}}} \tag{18}$$
$$-\sqrt{\frac{C - u_{22}a_2^2}{u_{11}}} - u_{12}a_2 \leq a_1 \leq -\sqrt{\frac{C - u_{22}a_2^2}{u_{11}}} - u_{12}a_2$$

After choosing the sphere radius and computing these bounds, we start searching the coefficient a_2 such that the bounds in (18) are satisfied and $[a_2]\bmod p \neq 0$. Then we search the coefficient a_1 meeting the boundaries requirements in (18) and satisfying $[a_1]\bmod p \neq 0$. Our algorithm is summarized below:

i) Perform Cholesky decomposition of $\mathbf{G} = \mathbf{R}^t\mathbf{R}$ and set $u_{ii} = R_{ii}^2, i = 1, 2$ and $u_{12} = \frac{R_{12}}{R_{11}}$.
ii) (*Initialization*) Set $i = 2, T_i = C, S_i = 0$.
iii) (*Compute bounds for a_i*) Set $Z = \sqrt{\frac{T_i}{u_{ii}}}, UB(a_i) = \lfloor Z - S_i \rfloor, LB(a_i) = \lceil -Z - S_i \rceil$ and set $a_i = LB(a_i) - 1$.
iv) (*Increase a_i*) Set $a_i = a_i + 1$, if $[a_i]\bmod p = 0$, go to $a_i = 1$. For $a_i \leq UB(a_i)$ go to vi), else go to v).
v) if $i = 2$ terminate, else set $i = i + 1$ and go to iv).
vi) (*Decrease i*) For $i = 1$ go to vii), else set $i = i - 1, S_1 = u_{12}a_2, T_1 = C - u_{22}a_2^2$ and go to step iii).
vii) (*Solution found*) For $\mathbf{a} = [0\ 0]^t$ terminate, else set $Q(\mathbf{a}) = C - T_1 + u_{11}(a_1 + S_1)^2$ and go to step iv).

B. Analog Network Coding

The processing at both phases is similar to the Gaussian channel case with two modifications: i) the scaling factor becomes $\beta = \sqrt{\frac{\rho}{1+\rho(h_1^2+h_2^2)}}$, and ii) knowledge of h_2 and h_1 respectively N_1 and N_2.

By deriving the computation of the effective SNR in the downlink phase, it is easy to show that the exchange rate of the ANC is equal to:

$$\mathcal{R}_{\text{ex,ANC}} = \min_{m=1,2} \left\{ \frac{1}{2} \log \left(1 + \frac{h_m^2 \rho^2}{1 + \rho(1+ \parallel \mathbf{h} \parallel^2)} \right) \right\} \quad (19)$$

C. Numerical results

En-to-end performance for the fading channel are depicted in Fig.3 and Fig.4.

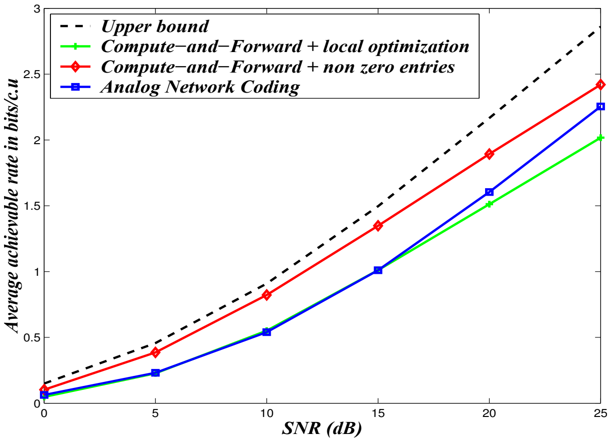

Fig. 3. Average exchange rate for the fading TWRC.

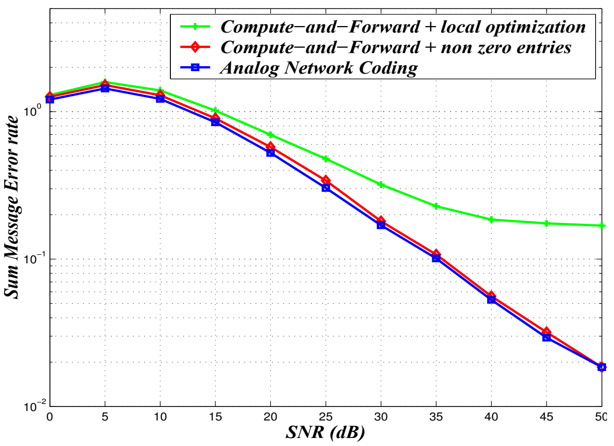

Fig. 4. Message Error rate for the fading TWRC.

First of all, numerical results confirm the noteworthy performance degradation when the non-zero condition in not respected. Efficiency and reliability of our proposed algorithm are as well showed. Our method brings a rate gain of about 0.5 bits/c.u and a gain of more than 15dB at high SNR values for fixed low MER over the CF based on local optimization criteria at the relay. Now compared to the ANC, we confirm the outperformance of the CF over this strategy when the exchange rate is considered. However, in terms of error rate, we report that both protocols achieve almost same performance unlike the Gaussian channel case. This can be explained by the following: In the Gaussian channel case the only factor that may impact the error performance is the noise which is totally eliminated under the CF framework. However, for the fading channel, both the channel gains approximation and the noise impact the error performance. We argue that the resulting behavior of the two protocols compared to the Gaussian channel is due to these two parameters.

VI. Conclusion

In this work we addressed a theoretical and numerical analysis of an end-to-end practical communication in the two way relay network scenario using two PLNC strategies: the Compute-and-Forward and the Analog Network Coding. For the Gaussian channel case, simulation results show that the CF outperforms the ANC in terms of both exchange rate and Sum error rate. For the fading channel, CF offers a rate gain over the ANC, however, their error rate performance are almost identical. We argue that this suboptimality compared to the Gaussian channel case is due to the channel quantization and to suboptimalilty of the lattice decoding at the relay.

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